Tractability of Multivariate Approximation Defined over Hilbert Spaces with Exponential Weights

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Abstract

We study multivariate approximation defined over tensor product Hilbert spaces. The domain space is a weighted tensor product Hilbert space with exponential weights which depend on two sequences $\boldsymbol{a} = \{a_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{b} = \{b_j\}_{j\in\mathbb{N}}$ of positive numbers, and on a bounded sequence of positive integers $\boldsymbol{m} = \{m_j\}_{j\in\mathbb{N}}$. The sequence $\boldsymbol{a}$ is non-decreasing and the sequence $\boldsymbol{b}$ is bounded from below by a positive number. We find necessary and sufficient conditions on $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{m}$ to achieve the standard and new notions of tractability in the worst case setting.

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1 Introduction

We approximate s-variate problems by algorithms that use finitely many linear functionals. The information complexity $n(\varepsilon, s)$ is defined as the minimal number of linear functionals which are needed to find an approximation to within an error threshold ε.

The standard notions of tractability deal with the characterization of s-variate problems for which the information complexity $n(\varepsilon, s)$ is *not* exponential in ε^{-1} and s. Since there are many different ways of measuring the lack of the exponential dependence we have various notions of tractability. For instance, weak tractability (WT) means that $\log\, n(\varepsilon, s)/(s + \varepsilon^{-1})$ goes to zero as $s + \varepsilon^{-1}$ approaches infinity, whereas quasi-polynomial tractability (QPT) means that $n(\varepsilon, s)$ can be bounded for all $s \in \mathbb{N}$ and all $\varepsilon \in (0, 1]$ by $C\, \exp(t\,(1 + \log\, s)(1 + \log\, \varepsilon^{-1}))$ for some C and t independent of both ε^{-1} and s. Analogously, we have polynomial tractability (PT) if $n(\varepsilon, s)$ can be bounded by a polynomial in ε^{-1} and s, and strong polynomial tractability (SPT) if $n(\varepsilon, s)$ can be bounded by a

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polynomial in ε^{-1} for all s. These notions of tractability have been extensively studied in many papers and the current state of the art in this field can be found in [12, 13, 14].

The notion of WT was recently refined in [18] by introducing (t_1, t_2)-weak tractability ((t_1, t_2)-WT) by assuming that $\log n(\varepsilon, s)/(s^{t_1} + \varepsilon^{-t_2})$ goes to zero as $s + \varepsilon^{-1}$ approaches infinity for some positive t_1 and t_2. Uniform weak tractability (UWT) was defined in [17] by assuming that (t_1, t_2)-WT holds for all $t_1, t_2 \in (0, 1]$. It is easy to check that for $t_1, t_2 \in (0, 1]$ we have the following hierarchy

$$\text{SPT} \Rightarrow \text{PT} \Rightarrow \text{QPT} \Rightarrow \text{UWT} \Rightarrow (t_1, t_2)\text{-WT} \Rightarrow \text{WT}.$$

All these standard notions are appropriate for s-variate problems for which the minimal errors are polynomially decaying. That is, for any $n \in \mathbb{N}$ we can find n linear functionals and an algorithm using these n linear functionals whose error decays like $\mathcal{O}(n^{-p})$ for some positive p and with the factor in the big $\mathcal{O}$ notation that may depend on s.

There is a stream of work with new notions of tractability which is relevant for s-variate problems for which the minimal errors are exponentially decaying, see [2, 3, 8, 10, 11, 15]. The new notions of tractability correspond to the standard notions of tractability but for the pair $(s, 1 + \log \varepsilon^{-1})$ instead of the pair (s, ε^{-1}). For instance the new notion of strong polynomial tractability means that we can bound $n(\varepsilon, s)$ by a polynomial in $1 + \log \varepsilon^{-1}$ for all $s \in \mathbb{N}$. Obviously, the new notions of tractability are more demanding than the standard ones. To distinguish them from the standard notions we add the prefix EC (exponential convergence) and we have EC-WT, EC-UWT, EC-(t_1, t_2)-WT, EC-QPT, EC-PT, and EC-SPT. For $t_1, t_2 \in (0, 1]$, we obviously have

$$\text{EC-SPT} \Rightarrow \text{EC-PT} \Rightarrow \text{EC-QPT} \Rightarrow \text{EC-UWT} \Rightarrow \text{EC-}(t_1, t_2)\text{-WT} \Rightarrow \text{EC-WT}.$$

We study (t_1, t_2)-WT and EC-(t_1, t_2)-WT for general positive t_1 and t_2, i.e., dropping the assumption that they are from $(0, 1]$. Obviously, if $t_1 > 1$ we do not have an exponential dependence on s^{t_1} but we may have the exponential dependence on s^{τ} for $\tau < t_1$. For $\tau = 1$, we may have an exponential dependence on s which is usually called the curse of of dimensionality. Nevertheless, the parameters t_1 and t_2 control the level of exponential behaviour with respect to s and ε^{-1}, and it seems to be an interesting problem to find the minimal, say, t_1 for which we have (t_1, t_2)-WT or EC-(t_1, t_2)-WT.

In this paper we study all these standard and new notions of tractability. This is done for general multivariate approximation defined over tensor product Hilbert spaces in the worst case setting. The construction of our problem is roughly as follows. For $s = 1$, we take a separable Hilbert space H of infinite dimension with an orthonormal basis $\{e_k\}_{k \in \mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and inner product $\langle \cdot, \cdot \rangle_H$. In general, we do not assume that H is a space of functions or that it is a reproducing kernel Hilbert space. Therefore we can only consider linear functionals as information used by algorithms.

From the space H, we construct a weighted Hilbert space in the following way. For given positive numbers a, b, ω with $\omega \in (0, 1)$, and a bounded sequence $\boldsymbol{m} = \{m_k\}_{k \in \mathbb{N}_0}$ of positive integers, the Hilbert space $H_{a,b}$ is a subspace of H for which $f \in H_{a,b}$ iff

$$\|f\|_{H_{a,b}} := \left(\sum_{j=0}^{m_0 - 1} |\langle f, e_j \rangle_H|^2 + \sum_{k=1}^{\infty} \omega^{-ak^b} \sum_{j=0}^{m_k - 1} \left| \left\langle f, e_{m_0 + \cdots + m_{k-1} + j} \right\rangle_H \right|^2 \right)^{1/2} < \infty.$$

Note that ω^{-ak^b} goes exponentially fast to infinity with k. Therefore, $\|f\|_{H_{a,b}} < \infty$ means that the sum of $\left| \left\langle f, e_{m_0 + \cdots + m_{k-1} + j} \right\rangle_H \right|^2$ for $j = 0, 1, \ldots, m_k - 1$ must decay exponentially fast with k.

The univariate approximation problem $\mathrm{APP}_1 : H_{a,b} \to H$ is defined as the embedding operator $\mathrm{APP}_1 f = f$. The s-variate approximation problem

$$\mathrm{APP}_s : H_{s,\boldsymbol{a},\boldsymbol{b}} \to H_s := \bigotimes_{j=1}^{s} H$$

is the embedding operator $\mathrm{APP}_s f = f$, where

$$H_{s,\boldsymbol{a},\boldsymbol{b}} := H_{a_1,b_1} \otimes H_{a_2,b_2} \otimes \cdots \otimes H_{a_s,b_s}$$

is the s-fold tensor product of the weighted spaces H_{a_j,b_j}. Here $\boldsymbol{a} = \{a_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{b} = \{b_j\}_{j\in\mathbb{N}}$. We assume that $a_1 > 0$, the a_j's are nondecreasing, and $\inf_j b_j > 0$.

The space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is a subset of H_s with exponentially decaying coefficients in the basis of H_s. The speed of the decay depends on the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ and ω of the problem.

Special instances of the spaces $H_{s,\boldsymbol{a},\boldsymbol{b}}$ are weighted Hermite and Korobov spaces which were already analyzed in the papers mentioned before. In fact, similarity in the analysis of weighted Hermite and Korobov spaces was an indication that more general weighted spaces can be also analyzed and it was the beginning of this paper. Other special instances of $H_{s,\boldsymbol{a},\boldsymbol{b}}$ are ℓ_2, cosine and Walsh spaces which have not been analyzed in this context before.

The weighted Hermite and Korobov spaces consist of *analytic* functions. This property is *not* shared, in general, for spaces $H_{s,\boldsymbol{a},\boldsymbol{b}}$. As for the univariate case, the space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ does not have to be a space of functions. But even if we assume that $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is a space of functions then the functions e_k do not have to be analytic or even smooth (for example this is the case for the Walsh space). It turns out that analyticity or smoothness of the functions e_k is irrelevant. Instead, the exponential decay of the coefficients in the basis of H_s is important. That is why the results for the space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ are similar to the results for the weighted Hermite and Korobov spaces.

We now briefly summarize the main results obtained in this paper. We first study when exponential convergence (EXP) and uniform exponential convergence (UEXP) hold. EXP holds if there is $q \in (0,1)$ such that for all $s \in \mathbb{N}$ we can find positive C_s, M_s, p_s for which the nth minimal worst case error for approximating APP_s, see Section 3, is bounded by

$$C_s\, q^{(n/M_s)^{p_s}} \quad \text{for all} \quad n \in \mathbb{N}.$$

The supremum of such p_s is called the exponent of EXP and denoted by p_s^*. UEXP holds if we can take $p_s = p > 0$ for all $s \in \mathbb{N}$, and the supremum of such p is called the exponent of UEXP and denoted by p^*.

We prove that EXP holds always with no extra conditions on the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}, \omega$, and $p_s^* = 1/\sum_{j=1}^{s} b_j^{-1}$, whereas UEXP holds iff $B := \sum_{j=1}^{\infty} b_j^{-1} < \infty$ and then $p^* = 1/B$. Hence, UEXP only requires that b_j^{-1}'s are summable and there are no extra conditions on the rest of the parameters.

We now turn to tractability. We obtain necessary and sufficient conditions on standard and new notions of tractability in terms of the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ and ω of the problems. Such conditions were not known before even for weighted Hermite or Korobov spaces. More precisely, UWT, QPT, EC-UWT, EC-QPT as well as (t_1,t_2)-WT and EC-(t_1,t_2)-WT were not studied before for the weighted Korobov spaces, and approximation has not been studied at all for Hermite spaces before.

To stress that we approximate APP_s, we denote the information complexity $n(\varepsilon, s)$ by $n(\varepsilon, \mathrm{APP}_s)$, see again Section 3. In this paper we present specific lower and upper bounds on $n(\varepsilon, \mathrm{APP}_s)$ from which we conclude various notions of standard and new tractability. We also present estimates of the tractability exponents. They are defined as the infimum of t for QPT and EC-QPT, or the infimum of the degree of polynomials in ε^{-1} for SPT and in $1 + \log \varepsilon^{-1}$ for EC-SPT which bound the information complexity $n(\varepsilon, \mathrm{APP}_s)$. We usually do not have the exact values of these exponents but only lower and upper bounds. It would be of interest to improve these bounds. In this section, we only mention when various tractability notions hold. We prove:

- (t_1, t_2)-WT holds for the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ and ω iff $t_1 > 1$ or $m_0 = 1$.
- EC-(t_1, t_2)-WT holds for the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ and ω iff $t_1 > 1$, or $t_2 > 1$ and $m_0 = 1$.
- WT holds iff $m_0 = 1$, whereas EC-WT holds iff
$$m_0 = 1 \quad \text{and} \quad \lim_{j \to \infty} a_j = \infty.$$
- UWT holds iff $m_0 = 1$, whereas EC-UWT holds iff
$$m_0 = 1 \quad \text{and} \quad \lim_{j \to \infty} \frac{\log\, a_j}{\log\, j} = \infty.$$
- QPT holds iff $m_0 = 1$, whereas EC-QPT holds iff
$$m_0 = 1, \quad \sup_{s \in \mathbb{N}} \frac{\sum_{j=1}^{s} b_j^{-1}}{1 + \log\, s} < \infty, \quad \text{and} \quad \liminf_{j \to \infty} \frac{(1 + \log\, j)\, \log\, a_j}{j} > 0.$$
- PT holds iff SPT holds iff[1]
$$m_0 = 1 \quad \text{and} \quad \lim_{j \to \infty} \frac{a_j}{\log j} > 0.$$
- EC-PT holds iff EC-SPT holds iff
$$m_0 = 1, \quad \sum_{j=1}^{\infty} b_j^{-1} < \infty, \quad \text{and} \quad \liminf_{j \to \infty} \frac{\log\, a_j}{j} > 0.$$

We would like to mention that the results for EC-UWT and EC-QPT for one particular example of $H_{s,a,b}$ (of Korobov type – see Example 3) were independently shown by Guiqiao Xu (private communication).

Observe that for $m_0 > 1$, only (t_1, t_2)-WT and EC-(t_1, t_2)-WT with $t_1 > 1$ hold. The reason is that
$$n(\varepsilon, \mathrm{APP}_s) \ge m_0^s \quad \text{for all } \varepsilon \in (0, 1),$$
and we have the curse of dimensionality. This also shows that the condition $t_1 > 1$ is sharp. So we have to assume that $m_0 = 1$ to obtain other notions of tractability in terms

[1] Under a simplifying assumption that the limit of $a_j / \log\, j$ exists.

of the conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$. Interestingly enough there are no conditions on m_k for $k > 0$ and on ω. However, the exponents of tractability as well as constants depend on m_k for $k > 0$ and on ω. We illustrate the necessary and sufficient conditions on various notions of tractability for $m_0 = 1$ and for

$$a_j = j^{\,v_1} \exp(v_2\, j) \quad \text{ and } \quad b_j = j^{\,v_3} \quad \text{for } j \geq 1$$

for some non-negative v_1, v_2 and v_3. Then

- EXP, (t_1, t_2)-WT, EC-(t_1, t_2)-WT with $t_1 > 1$, WT and QPT hold for all v_1, v_2, v_3,
- UEXP holds iff $v_3 > 1$,
- EC-WT, PT and SPT hold iff $v_1^2 + v_2^2 > 0$,
- EC-PT and EC-SPT hold iff $v_2 > 0$ and $v_3 > 1$.

The remaining sections of this paper are structured in the following way. We provide detailed information on the Hilbert spaces which are studied in the paper in Section 2. We outline the setting of the approximation problem in Section 3. The results on exponential and uniform exponential convergence are shown in Section 4. In Section 5 we prove the results on the various notions of tractability. A table which summarizes all conditions is presented in Section 6.

2 Weighted Hilbert Spaces

Let H be a separable Hilbert space over the real or complex field. To omit special cases, we also assume that H has infinite dimension. Let $\{e_k\}_{k\in\mathbb{N}_0}$ be its orthonormal basis, $\langle e_k, e_j \rangle_H = \delta_{k,j}$ for all $k, j \in \mathbb{N}_0$. Hence, for $f \in H$ one has

$$f = \sum_{k=0}^{\infty} \langle f, e_k \rangle_H \; e_k \quad \text{ with } \quad \sum_{k=0}^{\infty} |\, \langle f, e_k \rangle_H \,|^2 < \infty.$$

For $s \in \mathbb{N} := \{1, 2, \dots\}$, by $H_s = H \otimes H \otimes \cdots \otimes H$ we mean the s-fold tensor product of H. For $\boldsymbol{k} = [k_1, k_2, \dots, k_s] \in \mathbb{N}_0^s$, let $e_{\boldsymbol{k}} = e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_s}$. Clearly, $\{e_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^s}$ is an orthonormal basis of H_s and for $f \in H_s$ one has

$$f = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \langle f, e_{\boldsymbol{k}} \rangle_{H_s} \; e_{\boldsymbol{k}} \quad \text{ with } \quad \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} |\, \langle f, e_{\boldsymbol{k}} \rangle_{H_s} \,|^2 < \infty.$$

We now define a weighted Hilbert space which will depend on a number of parameters. Some of these parameters will be fixed while others will be varying. The fixed parameters are: a number $\omega \in (0, 1)$ and a bounded sequence $\boldsymbol{m} = \{m_k\}_{k\in\mathbb{N}_0}$ of positive integers. With the sequence $\boldsymbol{m}$ we associate a sequence $\boldsymbol{r} = \{r_k\}_{k\in\mathbb{N}_0}$ given by

$$\begin{aligned} r_0 &= 0, \\ r_k &= m_0 + m_1 + \cdots + m_{k-1} \quad \text{ for all } \; k \in \mathbb{N}. \end{aligned}$$

Clearly, $r_{k+1} = r_k + m_k \geq r_k + 1$. Furthermore,

$$\mathbb{N}_0 = \bigcup_{k=0}^{\infty} \{r_k, r_k + 1, \ldots, r_{k+1} - 1\}$$

and the sets $\{r_k, r_k + 1, \ldots, r_{k+1} - 1\}$ are disjoint.

The varying parameters are positive real numbers a and b. The weighted Hilbert space will be therefore denoted by $H_{a,b}$ and is defined as

$$H_{a,b} = \left\{ f \in H \ : \ \|f\|_{H_{a,b}} := \left(\sum_{k=0}^{\infty} \omega^{-ak^b} \sum_{j=r_k}^{r_{k+1}-1} |\left\langle f, e_j \right\rangle_H|^2 \right)^{1/2} < \infty \right\}.$$

As an example, consider $m_k \equiv 1$. Then $r_k = k$ and

$$\|f\|_{H_{a,b}} := \left(\sum_{k=0}^{\infty} \omega^{-ak^b} |\left\langle f, e_k \right\rangle_H|^2 \right)^{1/2}.$$

For a general $\boldsymbol{m}$, note that ω^{-ak^b} goes exponentially fast to infinity with k. Therefore $\|f\|_{H_{a,b}} < \infty$ means that $\sum_{j=r_k}^{r_{k+1}-1} |\left\langle f, e_j \right\rangle_H|^2$ must decay exponentially fast to zero as k goes to infinity.

The inner product in $H_{a,b}$ is given for $f, g \in H_{a,b}$ by

$$\left\langle f, g \right\rangle_{H_{a,b}} = \sum_{k=0}^{\infty} \omega^{-ak^b} \sum_{j=r_k}^{r_{k+1}-1} \left\langle f, e_j \right\rangle_H \overline{\left\langle g, e_j \right\rangle_H}.$$

Since $\omega^{-ak^b} \geq 1$, we have

$$\|f\|_H \leq \|f\|_{H_{a,b}} \quad \text{for all} \quad f \in H_{a,b}. \tag{1}$$

We now find an orthonormal basis $\{e_{n,a,b}\}_{n \in \mathbb{N}_0}$ of $H_{a,b}$. For $n \in \mathbb{N}_0$, there is a unique $k = k(n)$ such that $n \in \{r_{k(n)}, r_{k(n)} + 1, \ldots, r_{k(n)+1} - 1\}$. Then we set

$$e_{n,a,b} = \omega^{a[k(n)]^b/2} \, e_n.$$

We now verify that the sequence $\{e_{n,a,b}\}_{n \in \mathbb{N}_0}$ is orthonormal in $H_{a,b}$. Indeed, take $n_1, n_2 \in \mathbb{N}_0$. Then

$$\begin{aligned} \left\langle e_{n_1,a,b}, e_{n_2,a,b} \right\rangle_{H_{a,b}} &= \sum_{k=0}^{\infty} \omega^{-ak^b} \sum_{j=r_k}^{r_{k+1}-1} \left\langle e_{n_1,a,b}, e_j \right\rangle_H \overline{\left\langle e_{n_2,a,b}, e_j \right\rangle_H} \\ &= \sum_{k=0}^{\infty} \omega^{-ak^b + a[k(n_1)]^b/2 + a[k(n_2)]^b/2} \sum_{j=r_k}^{r_{k+1}-1} \left\langle e_{n_1}, e_j \right\rangle_H \overline{\left\langle e_{n_2}, e_j \right\rangle_H}. \end{aligned}$$

Suppose that $n_1 \neq n_2$. Then the last sum over j is zero for all $k \in \mathbb{N}_0$ due to the orthonormality of $\{e_j\}_{j \in \mathbb{N}_0}$. Suppose now that $n_1 = n_2$. Then the only non-zero term is for $k = k(n_1)$ and $j = n_1$, so that the sum is 1. Hence, $\left\langle e_{n_1,a,b}, e_{n_2,a,b} \right\rangle_{H_{a,b}} = \delta_{n_1,n_2}$.

Finally, note that $H_{a,b} \subseteq H = \mathrm{span}(e_1, e_2, \dots) = \mathrm{span}(e_{1,a,b}, e_{2,a,b}, \dots)$, which means that $\{e_{n,a,b}\}_{n\in\mathbb{N}_0}$ is an orthonormal basis of $H_{a,b}$, as claimed.

The norm in $H_{a,b}$ can now also be written as

$$\|f\|_{H_{a,b}} = \left(\sum_{n=0}^{\infty} |\langle f, e_{n,a,b}\rangle_{H_{a,b}}|^2 \right)^{1/2}.$$

We remark that $k(n) = 0$ for $n \in \{0, 1, \dots, m_0 - 1\}$ and therefore, $e_{n,a,b} = e_n$ and

$$\|e_{n,a,b}\|_{H_{a,b}} = \|e_n\|_H = 1 \quad \text{for all } \; n \in \{0, 1, \dots, m_0 - 1\}.$$

The last equality holds for m_0 elements, and $m_0 \geq 1$. This and (1) imply

$$\sup_{\|f\|_{H_{a,b}} \leq 1} \|f\|_H = 1. \tag{2}$$

Similarly as for the space H_s, we take the s-fold tensor products of the weighted space H_{a_j,b_j} with possibly different a_j and b_j such that

$$0 < a_1 \leq a_2 \leq \cdots \quad \text{and} \quad \inf_{j\in\mathbb{N}} b_j > 0. \tag{3}$$

That is,

$$H_{s,\boldsymbol{a},\boldsymbol{b}} = H_{a_1,b_1} \otimes H_{a_2,b_2} \otimes \cdots \otimes H_{a_s,b_s}.$$

For $\boldsymbol{n} = [n_1, n_2, \dots, n_s] \in \mathbb{N}_0^s$, define

$$e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} = e_{n_1,a_1,b_1} \otimes e_{n_2,a_2,b_2} \otimes \cdots \otimes e_{n_s,a_s,b_s}.$$

Then $\{e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}}\}_{\boldsymbol{n}\in\mathbb{N}_0^s}$ is an orthonormal basis of $H_{s,\boldsymbol{a},\boldsymbol{b}}$ and $f \in H_{s,\boldsymbol{a},\boldsymbol{b}}$ iff

$$f = \sum_{\boldsymbol{n}\in\mathbb{N}_0^s} \langle f, e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}}\rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} \, e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} \quad \text{with} \quad \|f\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}} := \left(\sum_{\boldsymbol{n}\in\mathbb{N}_0^s} |\,\langle f, e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}}\rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}}\,|^2 \right)^{1/2} < \infty.$$

We now show that

$$\|f\|_{H_s} \leq \|f\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}} \quad \text{for all } \; f \in H_{s,\boldsymbol{a},\boldsymbol{b}}. \tag{4}$$

Indeed, for $f \in H_{s,\boldsymbol{a},\boldsymbol{b}}$ we have $f = \sum_{\boldsymbol{n}\in\mathbb{N}_0^s} \alpha_{\boldsymbol{n}} \, e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}}$ with $\|f\|^2_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \sum_{\boldsymbol{n}\in\mathbb{N}_0^s} |\alpha_{\boldsymbol{n}}|^2 < \infty$. For any $n_j \in \mathbb{N}_0$ there is a unique $k(n_j) \in \mathbb{N}_0$ such that $n_j \in \{r_{k(n_j)}, r_{k(n_j)}+1, \dots, r_{k(n_j)+1} - 1\}$, and $e_{n_j,a_j,b_j} = \omega^{a_j [k(n_j)]^{b_j}/2} \, e_{n_j}$. Therefore

$$e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} = \left(\prod_{j=1}^{s} \omega^{a_j [k(n_j)]^{b_j}/2} \right) e_{\boldsymbol{n}}. \tag{5}$$

We have $f = \sum_{\boldsymbol{n}\in\mathbb{N}_0^s} \alpha_{\boldsymbol{n}} \left(\prod_{j=1}^{s} \omega^{a_j [k(n_j)]^{b_j}/2} \right) e_{\boldsymbol{n}}$ and

$$\|f\|_{H_s} = \left(\sum_{\boldsymbol{n}\in\mathbb{N}_0^s} |\alpha_{\boldsymbol{n}}|^2 \prod_{j=1}^{s} \omega^{a_j [k(n_j)]^{b_j}} \right)^{1/2} \leq \left(\sum_{\boldsymbol{n}\in\mathbb{N}_0^s} |\alpha_{\boldsymbol{n}}|^2 \right)^{1/2} = \|f\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}},$$

as claimed.

For $\boldsymbol{n} \in \{0, 1, \dots, m_0 - 1\}^s$, we have $k(n_j) = 0$ for $j = 1, 2, \dots, s$. Therefore $e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} = e_{\boldsymbol{n}}$ and

$$\|e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}}\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \|e_{\boldsymbol{n}}\|_{H_s} = 1 \quad \text{for all } \ \boldsymbol{n} \in \{0, 1, \dots, m_0 - 1\}^s.$$

The last equality holds for m_0^s elements. This and (4) imply

$$\sup_{\|f\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}} \le 1} \|f\|_{H_s} = 1. \tag{6}$$

Note that (5) implies that $\{e_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{N}_0^s}$ is orthogonal in $H_{s,\boldsymbol{a},\boldsymbol{b}}$ and

$$\|e_{\boldsymbol{n}}\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \prod_{j=1}^{s} \omega^{-a_j [k(n_j)]^{b_j}/2} \quad \text{for all } \ \boldsymbol{n} \in \mathbb{N}_0^s.$$

For $f \in H_s$ we have $f = \sum_{\boldsymbol{n} \in \mathbb{N}_0^s} \langle f, e_{\boldsymbol{n}} \rangle_{H_s} e_{\boldsymbol{n}}$ with $\sum_{\boldsymbol{n} \in \mathbb{N}_0^s} |\langle f, e_{\boldsymbol{n}} \rangle_{H_s}|^2 < \infty$. Such f belongs to $H_{s,\boldsymbol{a},\boldsymbol{b}}$ iff

$$\|f\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \left(\sum_{\boldsymbol{n} \in \mathbb{N}_0^s} \prod_{j=1}^{s} \omega^{-a_j [k(n_j)]^{b_j}} |\langle f, e_{\boldsymbol{n}} \rangle_{H_s}|^2 \right)^{1/2} < \infty.$$

As for the univariate case, we see that $\prod_{j=1}^{s} \omega^{-a_j [k(n_j)]^{b_j}}$ goes exponentially fast to infinity if one of the components of $\boldsymbol{n}$ goes to infinity. Therefore $|\langle f, e_{\boldsymbol{n}} \rangle_{H_s}|$ must decay exponentially fast to zero if one of the components of $\boldsymbol{n}$ approaches infinity.

Remark 1.

We stress that the spaces H, H_s and $H_{s,\boldsymbol{a},\boldsymbol{b}}$ do not have to be reproducing kernel Hilbert spaces, see [1] for general facts on reproducing kernel Hilbert spaces. Indeed, the initial space H does not have to be a function space. But if H is a Hilbert space of real or complex valued functions defined on, say, a common domain D, then it is well known that H is a reproducing kernel Hilbert space iff

$$\sum_{k=0}^{\infty} |e_k(x)|^2 < \infty \quad \text{for all } \ x \in D. \tag{7}$$

If (7) holds then

$$K(x, y) = \sum_{k=0}^{\infty} e_k(x)\, \overline{e_k(y)} \quad \text{for all } \ x, y \in D \tag{8}$$

is well-defined for all $x, y \in D$, because

$$|K(x, y)| = \left| \sum_{k=0}^{\infty} e_k(x) \overline{e_k(y)} \right| \le \left(\sum_{k=0}^{\infty} |e_k(x)|^2 \sum_{k=0}^{\infty} |e_k(y)|^2 \right)^{1/2} < \infty$$

by the Cauchy-Schwarz inequality. Moreover,

$$f(y) = \langle f, K(\cdot, y) \rangle_H \quad \text{for all } \ f \in H \text{ and } y \in D.$$

holds and so (8) is a reproducing kernel of H. If (7) holds then H_s is also a reproducing kernel Hilbert space and its kernel is

$$K_s(\boldsymbol{x},\boldsymbol{y}) = \prod_{j=1}^{s} K(x_j,y_j) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} e_{\boldsymbol{k}}(\boldsymbol{x})\,\overline{e_{\boldsymbol{k}}(\boldsymbol{y})} \quad \text{for all} \;\; \boldsymbol{x},\boldsymbol{y}\in D^s.$$

Similarly, the weighted space $H_{a,b}$ is a reproducing kernel Hilbert space iff

$$\sum_{k=0}^{\infty} \omega^{ak^b} \sum_{j=r_k}^{r_{k+1}-1} |e_j(x)|^2 < \infty \quad \text{for all} \;\; x\in D. \tag{9}$$

Clearly, the condition (9) is weaker than the condition (7). Hence, it may happen that H is not a reproducing kernel Hilbert space but $H_{a,b}$ is. We shall see examples of such spaces in a moment.

If (9) holds then the reproducing kernel of $H_{a,b}$ is

$$K_{a,b}(x,y) = \sum_{k=0}^{\infty} e_{k,a,b}(x)\overline{e_{k,a,b}(y)} = \sum_{k=0}^{\infty} \omega^{ak^b} \sum_{j=r_k}^{r_{k+1}-1} e_j(x)\overline{e_j(y)} \quad \text{for all} \;\; x,y\in D.$$

If (9) holds then $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is also a reproducing kernel Hilbert space and its kernel is

$$K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y}) = \prod_{j=1}^{s} K_{a_j,b_j}(x_j,y_j) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} e_{\boldsymbol{k},\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x})\,\overline{e_{\boldsymbol{k},\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{y})} \quad \text{for all} \;\; \boldsymbol{x},\boldsymbol{y}\in D^s.$$

□

We illustrate the weighted Hilbert spaces $H_{s,\boldsymbol{a},\boldsymbol{b}}$ by five examples.

Example 1. Weighted ℓ_2 Space

Let $H=\ell_2$ be the space of sequences in $\mathbb{C}$ with finite quadratic norm, i.e., $H=\ell_2=\{f:\mathbb{N}_0\to\mathbb{C}\,:\,\sum_{n=0}^{\infty}|f(n)|^2<\infty\}$. Let e_k be the k-th canonical element $e_k(n)=\delta_{k,n}$.

Then $H_s=\{f\,:\,\mathbb{N}_0^s\to\mathbb{C}\,:\,\sum_{\boldsymbol{n}\in\mathbb{N}_0^s}|f(\boldsymbol{n})|^2<\infty\}$. For $\boldsymbol{k}=[k_1,k_2,\dots,k_s]\in\mathbb{N}_0^s$, let $e_{\boldsymbol{k}}(\boldsymbol{n})=\prod_{j=1}^{s}e_{k_j}(n_j)$ for all $\boldsymbol{n}=[n_1,n_2,\dots,n_s]\in\mathbb{N}_0^s$. The inner product in H_s is $\langle f,g\rangle_{H_s}=\sum_{\boldsymbol{k}\in\mathbb{N}_0^s}f(\boldsymbol{k})g(\boldsymbol{k})$. Note that $\langle e_{\boldsymbol{k}_1},e_{\boldsymbol{k}_2}\rangle_{H_s}=\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ and $\sum_{\boldsymbol{k}\in\mathbb{N}_0^s}|e_{\boldsymbol{k}}(\boldsymbol{n})|^2=1$. Hence, H_s is a reproducing kernel Hilbert space with kernel function

$$K_s(\boldsymbol{l},\boldsymbol{n}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} e_{\boldsymbol{k}}(\boldsymbol{l})e_{\boldsymbol{k}}(\boldsymbol{n}) = \delta_{\boldsymbol{l},\boldsymbol{n}} \quad \text{for } \boldsymbol{l},\boldsymbol{n}\in\mathbb{N}_0^s.$$

For $H_{s,\boldsymbol{a},\boldsymbol{b}}$, we take $m_k\equiv 1$. Then $r_k=k$ and $k(n)=n$. The inner product of $H_{s,\boldsymbol{a},\boldsymbol{b}}$ for $f,g\in H_{s,\boldsymbol{a},\boldsymbol{b}}$ is given by

$$\langle f,g\rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \omega^{-\sum_{j=1}^{s}a_jk_j^{b_j}}\, f(\boldsymbol{k})\,g(\boldsymbol{k}).$$

Hence $f\in H_{s,\boldsymbol{a},\boldsymbol{b}}$ means that the $|f(\boldsymbol{k})|$ of f decrease exponentially fast. $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is a reproducing kernel Hilbert space with kernel

$$K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{l},\boldsymbol{n}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \omega^{\sum_{j=1}^{s}a_jk_j^{b_j}}\, e_{\boldsymbol{k}}(\boldsymbol{l})e_{\boldsymbol{k}}(\boldsymbol{n}).$$

□

Example 2. Weighted Hermite Space

Let $\rho(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ for all $x \in \mathbb{R}$ be the Gaussian weight in the real line and let Her_k be the Hermite polynomial of degree k,

$$\mathrm{Her}_k(x) = \frac{(-1)^k}{\sqrt{k!}} \exp(x^2/2) \frac{\mathrm{d}^k}{\mathrm{d}x^k} \exp(-x^2/2) \quad \text{for all} \;\; x \in \mathbb{R}.$$

Now we consider the Hilbert space of real functions which are Lebesgue square-integrable with respect to ρ and have an absolutely convergent Hermite series, i.e.,

$$H = \Big\{ f : \mathbb{R} \to \mathbb{R} \,:\, f \text{ measurable}, \int_{\mathbb{R}} |f(x)|^2 \rho(x)\, \mathrm{d}x < \infty, \\ f(x) = \sum_{k \in \mathbb{N}_0} \widehat{f}_k \mathrm{Her}_k(x) \text{ absolutely convergent} \Big\}$$

with inner product $\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) \rho(x)\, \mathrm{d}x$. Since it is known that $\{\mathrm{Her}_k\}_{k \in \mathbb{N}_0}$ is orthonormal, we can take $e_k = \mathrm{Her}_k$. Clearly, H is *not* a reproducing kernel Hilbert space. Then

$$H_s = \Big\{ f : \mathbb{R}^s \to \mathbb{R} \,:\, f \text{ measurable}, \int_{\mathbb{R}^s} |f(\boldsymbol{x})|^2 \rho_s(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} < \infty, \\ f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}_{\boldsymbol{k}} \mathrm{Her}_{\boldsymbol{k}}(\boldsymbol{x}) \text{ absolutely convergent} \Big\}$$

with

$$\rho_s(\boldsymbol{x}) = \frac{1}{(2\pi)^{s/2}} \exp\left(-\frac{1}{2} \sum_{j=1}^s x_j^2 \right) \qquad \text{for all} \;\; \boldsymbol{x} = [x_1, x_2, \dots, x_s] \in \mathbb{R}^s.$$

For $\boldsymbol{k} \in \mathbb{N}_0^s$, we take $e_{\boldsymbol{k}}(\boldsymbol{x}) = \mathrm{Her}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^s \mathrm{Her}_{k_j}(x_j)$ for all $\boldsymbol{x} \in \mathbb{R}^s$. Then $\{e_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{N}_0^s}$ is an orthonormal basis of H_s. Obviously, H_s is *not* a reproducing kernel Hilbert space for any $s \in \mathbb{N}$.

The weighted Hermite space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is obtained by taking $m_k \equiv 1$. Then $r_k = k$ and $k(n) = n$. The inner product of $H_{s,\boldsymbol{a},\boldsymbol{b}}$ for $f, g \in H_{s,\boldsymbol{a},\boldsymbol{b}}$ is given by

$$\langle f, g \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \omega^{-\sum_{j=1}^s a_j k_j^{b_j}} \, \widehat{f}_{\boldsymbol{k}} \, \widehat{g}_{\boldsymbol{k}},$$

where $\widehat{f}_{\boldsymbol{k}}$ and $\widehat{g}_{\boldsymbol{k}}$ denote the $\boldsymbol{k}$th Hermite coefficients of f and g,

$$\widehat{f}_{\boldsymbol{k}} = \langle f, \mathrm{Her}_{\boldsymbol{k}} \rangle_{L_2(\mathbb{R}^s, \rho_s)} = \int_{\mathbb{R}^s} f(\boldsymbol{x})\, \mathrm{Her}_{\boldsymbol{k}}(\boldsymbol{x})\, \rho_s(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \text{for all} \;\; \boldsymbol{k} \in \mathbb{N}_0^s.$$

The weighted Hermite space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is a reproducing kernel Hilbert space due to Cramer's bound which states that

$$|\mathrm{Her}_k(x)| \le (2\pi)^{1/4} \exp(x^2/4) \quad \text{for all} \quad x \in \mathbb{R} \;\; \text{and} \;\; k \in \mathbb{N}_0,$$

see [16, p. 324]. Indeed, this bound leads to

$$\sum_{\boldsymbol{k} \in \mathbb{N}_0^s} [e_{\boldsymbol{k},\boldsymbol{a},\boldsymbol{b}}(x)]^2 = \prod_{j=1}^s \sum_{k=0}^\infty \omega^{a_j k^{b_j}} \, [\mathrm{Her}_k(x_j)]^2 \le \prod_{j=1}^s (2\pi)^{1/2} \exp(x_j^2/2) \sum_{k=0}^\infty \omega^{a_j k^{b_j}} < \infty$$

since the series $\sum_{k=0}^{\infty} \omega^{a_j k^{b_j}} < \infty$ for all positive a_j and b_j. The reproducing kernel of $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is

$$K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \omega^{\sum_{j=1}^s a_j k^{b_j}} \operatorname{Her}_{\boldsymbol{k}}(\boldsymbol{x}) \operatorname{Her}_{\boldsymbol{k}}(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^s.$$

More information on weighted Hermite spaces can be found in [8, 9]. □

Example 3. Weighted Korobov Space

We now take H as the Hilbert space of complex-valued, square-integrable functions on $[0,1]$ with absolutely convergent Fourier series, i.e.,

$$H = \Big\{ f : [0,1] \to \mathbb{C} \,:\, f \text{ measurable}, \int_0^1 |f(x)|^2 \,\mathrm{d}x < \infty, \\ f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}_k \exp(2\pi \mathtt{i} k x) \text{ absolutely convergent} \Big\}$$

with inner product $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x$. The orthonormal basis $\{e_k\}_{k\in\mathbb{N}_0}$ of H is taken as

$$e_0(x) = 1, \quad e_{2k-1}(x) = \exp(2\pi\mathtt{i}k\,x), \quad e_{2k}(x) = \exp(-2\pi\mathtt{i}k\,x),$$

for $k \in \mathbb{N}$ with $\mathtt{i} = \sqrt{-1}$. Then

$$H_s = \Big\{ f : [0,1]^s \to \mathbb{C} \,:\, f \text{ measurable}, \int_{[0,1]^s} |f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} < \infty, \\ f(x) = \sum_{\boldsymbol{k}\in\mathbb{Z}} \widehat{f}_{\boldsymbol{k}} \exp(2\pi \mathtt{i} \boldsymbol{k} \boldsymbol{x}) \text{ absolutely convergent} \Big\}$$

and $\{e_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^s}$ with

$$e_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^s e_{k_j}(x_j) \quad \text{for all } \boldsymbol{x}\in[0,1]^s$$

as its orthonormal basis. Then $f \in H_s$ iff

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \widehat{f}_{\boldsymbol{h}} \exp\left(2\pi\mathtt{i}\,\boldsymbol{h}\cdot\boldsymbol{x}\right) \quad \text{for all } \boldsymbol{x}\in[0,1]^s$$

with $\sum_{\boldsymbol{h}\in\mathbb{Z}^s} |\widehat{f}_{\boldsymbol{h}}|^2 < \infty$, where $\boldsymbol{h}\cdot\boldsymbol{x}$ denotes the usual dot product. Here, $\mathbb{Z}$ is the set of all integers, $\mathbb{Z} := \{\dots,-1,0,1,\dots\}$, and

$$\widehat{f}_{\boldsymbol{h}} = \langle f, e_{\boldsymbol{h}}\rangle_{L_2} = \int_{[0,1]^s} f(\boldsymbol{x}) \exp(-2\pi\mathtt{i}\,\boldsymbol{h}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

is the $\boldsymbol{h}$th Fourier coefficient. Clearly, H_s is *not* a reproducing kernel Hilbert space for all $s \in \mathbb{N}$.

The weighted Korobov space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is obtained by taking $m_0 = 1$ and $m_k = 2$ for all $k \in \mathbb{N}$. Then $r_0 = 0$ and $r_k = 2k-1$ for all $k \in \mathbb{N}$. The inner product of $H_{s,\boldsymbol{a},\boldsymbol{b}}$ for $f, g \in H_{s,\boldsymbol{a},\boldsymbol{b}}$ is given by

$$\langle f, g\rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \omega^{-\sum_{j=1}^s a_j |h_j|^{b_j}} \,\widehat{f}_{\boldsymbol{h}}\, \overline{\widehat{g}_{\boldsymbol{h}}}.$$

The space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is a reproducing kernel Hilbert space and its reproducing kernel is

$$K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \omega^{\sum_{j=1}^s a_j|h_j|^{b_j}} \ \exp(2\pi\mathtt{i}\,\boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})) \quad \text{for all } \ \boldsymbol{x},\boldsymbol{y}\in[0,1]^s.$$

The weighted Korobov space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is a space of periodic functions with period 1 for each variable. More information on these spaces can be found in [2, 3, 10, 11]. □

Example 4. Weighted Cosine Space

We take H as the Hilbert space of real-valued and square integrable functions defined on $[0,1]$ with absolutely convergent cosine series, more precisely,

$$H = \Big\{ f:[0,1]\to\mathbb{R} \,:\, f \text{ measurable}, \int_0^1 |f(x)|^2\,\mathrm{d}x < \infty,$$
$$f(x) = \widehat{f}_0 + \sum_{k\in\mathbb{N}} \widehat{f}_k \sqrt{2}\cos(\pi k x) \text{ absolutely convergent} \Big\}$$

The orthonormal basis $\{e_k\}_{k\in\mathbb{N}_0}$ of H is then taken as

$$e_0(x) = 1, \ \text{and} \quad e_k(x) = \sqrt{2}\cos(\pi k x) \ \text{ for } \ k\in\mathbb{N}.$$

Then we take, as in the previous examples, $H_s = H\otimes\cdots\otimes H$ and $\{e_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^s}$ with

$$e_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^s e_{k_j}(x_j) \quad \text{for all } \ \boldsymbol{x}\in[0,1]^s$$

as its orthonormal basis. For $\boldsymbol{h} = [h_1,h_2,\dots,h_s]\in\mathbb{N}_0^s$ we denote by $|\boldsymbol{h}|_0$ the number of indices $j\in\{1,2,\dots,s\}$ for which $h_j\neq 0$. Then $f\in H_s$ iff

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in\mathbb{N}_0^s} \widetilde{f}_{\boldsymbol{h}} (\sqrt{2})^{|\boldsymbol{h}|_0} \left(\prod_{j=1}^s \cos(\pi h_j x_j)\right) \qquad \text{for all } \ \boldsymbol{x}\in[0,1]^s$$

with $\sum_{\boldsymbol{h}\in\mathbb{N}_0^s} |\widetilde{f}_{\boldsymbol{h}}|^2 < \infty$. Here

$$\widetilde{f}_{\boldsymbol{h}} = \langle f, e_{\boldsymbol{h}}\rangle_{L_2} = \int_{[0,1]^s} f(x)(\sqrt{2})^{|\boldsymbol{h}|_0} \left(\prod_{j=1}^s \cos(\pi h_j x_j)\right) \mathrm{d}\boldsymbol{x}$$

is the $\boldsymbol{h}$th cosine coefficient. Clearly, H_s is *not* a reproducing kernel Hilbert space for all $s\in\mathbb{N}$.

The weighted cosine space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is obtained by taking $m_k\equiv 1$. Then $r_k = k$ and $k(n) = n$. The inner product of $H_{s,\boldsymbol{a},\boldsymbol{b}}$ for $f,g\in H_{s,\boldsymbol{a},\boldsymbol{b}}$ is given by

$$\langle f,g\rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \sum_{\boldsymbol{h}\in\mathbb{N}_0^s} \omega^{-\sum_{j=1}^s a_j|h_j|^{b_j}} \ \widetilde{f}_{\boldsymbol{h}}\,\widetilde{g}_{\boldsymbol{h}}.$$

The space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is a reproducing kernel Hilbert space and its reproducing kernel is

$$K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{N}_0^s} \omega^{\sum_{j=1}^s a_j|h_j|^{b_j}}\, 2^{|\boldsymbol{h}|_0} \left(\prod_{j=1}^s \cos(\pi h_j x_j)\cos(\pi h_j y_j)\right) \qquad \text{for all } \ \boldsymbol{x},\boldsymbol{y}\in[0,1]^s.$$

More information on cosine spaces with finite smoothness can be found in [4]. □

Example 5. Weighted Walsh Space

Let us denote by wal_k the kth Walsh function in some fixed integer base $b \geq 2$, see for example [6, Appendix A] for further details.

We now study the Hilbert space of complex-valued, square-integrable functions on $[0,1]$ with absolutely convergent Walsh series, i.e.,

$$H = \Big\{ f : [0,1] \to \mathbb{C} \,:\, f \text{ measurable}, \int_0^1 |f(x)|^2 \,\mathrm{d}x < \infty, \\ f(x) = \sum_{k \in \mathbb{N}_0} \widehat{f}_k \mathrm{wal}_k(x) \text{ absolutely convergent} \Big\}.$$

For this example the orthonormal basis $\{e_k\}_{k\in\mathbb{N}_0}$ of H is taken as

$$e_k(x) = \mathrm{wal}_k(x) \quad \text{for all} \;\; k \in \mathbb{N}_0.$$

Then $H_s = L_2([0,1]^s)$

$$H_s = \Big\{ f : [0,1]^s \to \mathbb{R} \,:\, f \text{ measurable}, \int_{[0,1]^s} |f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} < \infty, \\ f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}_{\boldsymbol{k}} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \text{ absolutely convergent} \Big\}.$$

and $\{e_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^s}$ with

$$e_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^s e_{k_j}(x_j) = \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^s \mathrm{wal}_{k_j}(x_j) \quad \text{for all} \;\; \boldsymbol{x} \in [0,1]^s$$

as its orthonormal basis. Then $f \in H_s$ iff

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h} \in \mathbb{N}_0^s} \widehat{f}_{\boldsymbol{h},\mathrm{wal}} \, \mathrm{wal}_{\boldsymbol{h}}(\boldsymbol{x}) \quad \text{for all} \;\; \boldsymbol{x} \in [0,1]^s$$

with $\sum_{\boldsymbol{h}\in\mathbb{N}_0^s} |\widehat{f}_{\boldsymbol{h},\mathrm{wal}}|^2 < \infty$. Here,

$$\widehat{f}_{\boldsymbol{h},\mathrm{wal}} = \langle f, \mathrm{wal}_{\boldsymbol{h}} \rangle_{L_2} = \int_{[0,1]^s} f(\boldsymbol{x}) \, \overline{\mathrm{wal}_{\boldsymbol{h}}(\boldsymbol{x})} \,\mathrm{d}\boldsymbol{x}$$

is the $\boldsymbol{h}$th Walsh coefficient.

The weighted Walsh space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is obtained by taking $m_k \equiv 1$. Then $r_k = k$ and $k(n) = n$. The inner product of $H_{s,\boldsymbol{a},\boldsymbol{b}}$ for $f, g \in H_{s,\boldsymbol{a},\boldsymbol{b}}$ is given by

$$\langle f, g \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \sum_{\boldsymbol{h} \in \mathbb{N}_0^s} \omega^{-\sum_{j=1}^s a_j |h_j|^{b_j}} \, \widehat{f}_{\boldsymbol{h},\mathrm{wal}} \, \overline{\widehat{g}_{\boldsymbol{h},\mathrm{wal}}}.$$

The space $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is a reproducing kernel Hilbert space and its reproducing kernel is

$$K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h} \in \mathbb{N}_0^s} \omega^{\sum_{j=1}^s a_j |h_j|^{b_j}} \, \mathrm{wal}_{\boldsymbol{h}}(\boldsymbol{x}) \, \overline{\mathrm{wal}_{\boldsymbol{h}}(\boldsymbol{y})} \quad \text{for all} \;\; \boldsymbol{x}, \boldsymbol{y} \in [0,1]^s.$$

More information on the Walsh spaces with finite smoothness can be found in [5, 6]. □

3 Multivariate Approximation

By multivariate approximation we mean an embedding operator $\mathrm{APP}_s : H_{s,\boldsymbol{a},\boldsymbol{b}} \to H_s$ given by

$$\mathrm{APP}_s f = f \quad \text{for all } \ f \in H_{s,\boldsymbol{a},\boldsymbol{b}}.$$

Due to (6), the operator APP_s is well defined, and it is a continuous linear operator. Furthermore, $\|\mathrm{APP}_s f\|_{H_s} \le \|f\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}}$ for all $f \in H_{s,\boldsymbol{a},\boldsymbol{b}}$ and

$$\|\mathrm{APP}_s\| = 1 \quad \text{for all } \ s \in \mathbb{N}.$$

We will later show that APP_s is a compact operator.

We want to approximate $\mathrm{APP}_s f$ by algorithms $A_n : H_{s,\boldsymbol{a},\boldsymbol{b}} \to H_s$ that use at most n continuous linear functionals of f. Without loss of generality, see e.g. [12, 19], we may restrict ourselves to linear algorithms of the form

$$A_n f = \sum_{j=1}^{n} L_j(f)\, g_j \quad \text{for all } \ f \in H_{s,\boldsymbol{a},\boldsymbol{b}}$$

for some $L_j \in H^*_{s,\boldsymbol{a},\boldsymbol{b}}$ and $g_j \in H_s$ for $j = 1, 2, \dots, n$.

We consider the worst case setting in which the error of A_n is defined as

$$e(A_n) = \sup_{\|f\|_{H_{s,\boldsymbol{a},\boldsymbol{b}}} \le 1} \|\mathrm{APP}_s f - A_n f\|_{H_s} = \|\mathrm{APP}_s - A_n\|.$$

For $n = 0$, we have the so-called initial error which is achieved by the zero algorithm $A_0 = 0$, and $e(A_0) = \|\mathrm{APP}_s\| = 1$.

By the nth minimal (worst case) error we mean the minimal error among all algorithms A_n,

$$e(n, \mathrm{APP}_s) = \inf_{A_n} e(A_n).$$

Clearly, $e(0, \mathrm{APP}_s) = 1$. In a moment an algorithm A^*_n for which the infimum is attained will be presented.

By the information complexity $n(\varepsilon, \mathrm{APP}_s)$ we mean the minimal n for which we can find an algorithm A_n with error at most $\varepsilon \in (0, \infty)$,

$$n(\varepsilon, \mathrm{APP}_s) = \min\{\, n \ : \ e(n, \mathrm{APP}_s) \le \varepsilon \,\}.$$

Clearly, $n(\varepsilon, \mathrm{APP}_s) = 0$ for all $\varepsilon \ge 1$, and therefore the only ε's of interest are from $(0, 1)$.

It is well known, see again e.g., [12, 19], that the nth minimal errors $e(n, \mathrm{APP}_s)$ and the information complexity $n(\varepsilon, \mathrm{APP}_s)$ depend on the eigenvalues of the continuous and linear operator $W_s = \mathrm{APP}^*_s \mathrm{APP}_s : H_{s,\boldsymbol{a},\boldsymbol{b}} \to H_{s,\boldsymbol{a},\boldsymbol{b}}$. The operator W_s is self-adjoint and in a moment we shall see that W_s is also compact. Let $(\lambda_{s,j}, \eta_{s,j})$ be the eigenpairs of W_s,

$$W_s \eta_{s,j} = \lambda_{s,j}\, \eta_{s,j} \quad \text{for all } \ j \in \mathbb{N},$$

where the eigenvalues $\lambda_{s,j}$ are ordered,

$$\lambda_{s,1} \ge \lambda_{s,2} \ge \cdots \ge 0,$$

and the eigenelements $\eta_{s,j}$ are orthonormal,

$$\langle \eta_{s,j_1}, \eta_{s,j_2} \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \delta_{j_1,j_2} \quad \text{for all} \;\; j_1, j_2 \in \mathbb{N}.$$

Then the nth minimal error is attained for the algorithm

$$A_n^* f = \sum_{j=1}^{n} \langle f, \eta_{s,j} \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} \, \eta_{s,j} \quad \text{for all} \;\; f \in H_{s,\boldsymbol{a},\boldsymbol{b}},$$

and

$$e(n, \mathrm{APP}_s) = e(A_n^*) = \sqrt{\lambda_{s,n+1}} \quad \text{for all} \;\; n \in \mathbb{N}_0.$$

This implies that the information complexity is equal to

$$n(\varepsilon, \mathrm{APP}_s) = \min\{\, n \in \mathbb{N}_0 \; : \; \lambda_{s,n+1} \le \varepsilon^2 \,\}. \tag{10}$$

We now find the eigenpairs of W_s. Using the notation and results of the previous section, we know that $\{e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}}\}_{\boldsymbol{n}\in\mathbb{N}_0^s}$ is an orthonormal basis of $H_{s,\boldsymbol{a},\boldsymbol{b}}$. We prove that

$$W_s e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} = \omega^{\sum_{j=1}^s a_j [k(n_j)]^{b_j}} \, e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} \quad \text{for all} \;\; \boldsymbol{n} \in \mathbb{N}_0^s.$$

Indeed, for $f, g \in H_{s,\boldsymbol{a},\boldsymbol{b}}$ we have

$$\langle \mathrm{APP}_s f, \mathrm{APP}_s g \rangle_{H_s} = \langle f, \mathrm{APP}_s^* \mathrm{APP}_s g \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \langle f, W_s g \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}}.$$

Taking $f = e_{\boldsymbol{n}_1,\boldsymbol{a},\boldsymbol{b}}$ and $g = e_{\boldsymbol{n}_2,\boldsymbol{a},\boldsymbol{b}}$ for arbitrary $\boldsymbol{n}_1, \boldsymbol{n}_2 \in \mathbb{N}_0^s$ we obtain from (5),

$$\begin{aligned}
\langle e_{\boldsymbol{n}_1,\boldsymbol{a},\boldsymbol{b}}, W_s e_{\boldsymbol{n}_2,\boldsymbol{a},\boldsymbol{b}} \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} &= \langle e_{\boldsymbol{n}_1,\boldsymbol{a},\boldsymbol{b}}, e_{\boldsymbol{n}_2,\boldsymbol{a},\boldsymbol{b}} \rangle_{H_s} \\
&= \left(\prod_{j=1}^s \omega^{a_j [k((n_1)_j)]^{b_j}/2} \, \omega^{a_j [k((n_2)_j)]^{b_j}/2} \right) \langle e_{\boldsymbol{n}_1}, e_{\boldsymbol{n}_2} \rangle_{H_s} \\
&= \omega^{\sum_{j=1}^s a_j [k((n_1)_j)]^{b_j}/2 + a_j [k((n_2)_j)]^{b_j}/2} \, \delta_{\boldsymbol{n}_1,\boldsymbol{n}_2}.
\end{aligned}$$

Hence,

$$\langle e_{\boldsymbol{n}_1,\boldsymbol{a},\boldsymbol{b}}, W_s e_{\boldsymbol{n}_2,\boldsymbol{a},\boldsymbol{b}} \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = 0 \quad \text{for all} \;\; \boldsymbol{n}_1 \ne \boldsymbol{n}_2,$$

and

$$\langle e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}}, W_s e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} = \omega^{\sum_{j=1}^s a_j [k(n_j)]^{b_j}}.$$

This means that

$$W_s e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} = \sum_{\boldsymbol{n}_1 \in \mathbb{N}_0^s} \langle W_s e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}}, e_{\boldsymbol{n}_1,\boldsymbol{a},\boldsymbol{b}} \rangle_{H_{s,\boldsymbol{a},\boldsymbol{b}}} \, e_{\boldsymbol{n}_1,\boldsymbol{a},\boldsymbol{b}} = \omega^{\sum_{j=1}^s a_j [k(n_j)]^{b_j}} \, e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}},$$

as claimed. Hence,

$$\left(\omega^{\sum_{j=1}^s a_j [k(n_j)]^{b_j}}, e_{\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}} \right)_{\boldsymbol{n}\in\mathbb{N}_0^s}$$

are the eigenpairs of W_s.

As an example consider the weighted Hermite space or the weighted cosine space for which $m_k \equiv 1$. Then $k(n_j) = n_j$ and the eigenpairs are of the form

$$\left(\omega^{\sum_{j=1}^s a_j n_j^{b_j}}, \omega^{\sum_{j=1}^s a_j n_j^{b_j}/2} \, e_{\boldsymbol{n}} \right)_{\boldsymbol{n}\in\mathbb{N}_0^s}.$$

For the weighted Korobov space, we have $m_0 = 1$ and $m_k = 2$ for all $k \in \mathbb{N}$. Then $k(n_j) = \lceil n_j/2 \rceil$ and the eigenpairs are of the form

$$\left(\omega^{\sum_{j=1}^s a_j \lceil n_j/2 \rceil^{b_j}}, \omega^{\sum_{j=1}^s a_j \lceil n_j/2 \rceil^{b_j}/2} e_{\boldsymbol{n}} \right)_{\boldsymbol{n} \in \mathbb{N}_0^s}.$$

We turn to the general case. The eigenvalues of W_s may be multiple. Indeed, for $n_j \in \mathbb{N}_0$ we obtain the same $k(n_j)$ for all $n_j \in \{r_{k(n_j)}, r_{k(n_j)} + 1, \ldots, r_{k(n_j)+1} - 1\}$, i.e., for $r_{k(n_j)+1} - r_{k(n_j)} = m_{k(n_j)}$ different values of n_j. This means that W_s has the eigenvalues

$$\omega^{\sum_{j=1}^s a_j k_j^{b_j}} \quad \text{of multiplicity} \quad m_{\boldsymbol{k}} := \sum_{\boldsymbol{l} \in \mathcal{B}_{\boldsymbol{k}}} m_{l_1} m_{l_2} \cdots m_{l_s},$$

where $\mathcal{B}_{\boldsymbol{k}} = \{\boldsymbol{l} \in \mathbb{N}_0^s \,:\, \sum_{j=1}^s a_j l_j^{b_j} = \sum_{j=1}^s a_j k_j^{b_j}\}$. In particular, the largest eigenvalue $\lambda_{s,1} = 1$, obtained for $k_j = 0$ for all $j = 1, 2, \ldots, s$, has multiplicity m_0^s. So for $m_0 = 1$ the largest eigenvalue is single.

Clearly, the sequence of ordered eigenvalues $\{\lambda_{s,j}\}_{j \in \mathbb{N}}$ is the same as the sequence $\{\omega^{\sum_{j=1}^s a_j [k(n_j)]^{b_j}}\}_{\boldsymbol{n} \in \mathbb{N}_0^s}$. Furthermore it is obvious that $\lim_{j \to \infty} \lambda_{s,j} = 0$, which implies that APP_s as well as W_s are compact.

We now find a more convenient formula for the information complexity $n(\varepsilon, \mathrm{APP}_s)$. From (10) we conclude that for $\varepsilon \in (0, \infty)$ we have

$$n(\varepsilon, \mathrm{APP}_s) = |\{\, j \in \mathbb{N}_0 \,:\, \lambda_{s,j} > \varepsilon^2 \,\}|,$$

or equivalently

$$n(\varepsilon, \mathrm{APP}_s) = |\{\, j \in \mathbb{N}_0 \,:\, \log \lambda_{s,j}^{-1} < \log \varepsilon^{-2} \,\}|.$$

All eigenvalues $\lambda_{s,j}$ are of the form $\omega^{\sum_{j=1}^s a_j k_j^{b_j}}$ with multiplicity $m_{\boldsymbol{k}}$ for $\boldsymbol{k} \in \mathbb{N}_0^s$. Therefore

$$\log \omega^{-\sum_{j=1}^s a_j k_j^{b_j}} = \left(\sum_{j=1}^s a_j k_j^{b_j} \right) \log \omega^{-1}$$

and

$$\log \omega^{-\sum_{j=1}^s a_j k_j^{b_j}} < \log \varepsilon^{-2} \quad \text{iff} \quad \sum_{j=1}^s a_j k_j^{b_j} < \frac{\log \varepsilon^{-2}}{\log \omega^{-1}}.$$

Let

$$A(\varepsilon, s) = \left\{ \boldsymbol{k} \in \mathbb{N}_0^s \,:\, \sum_{j=1}^s a_j k_j^{b_j} < \frac{\log \varepsilon^{-2}}{\log \omega^{-1}} \right\}.$$

Then

$$n(\varepsilon, \mathrm{APP}_s) = \sum_{\boldsymbol{k} \in A(\varepsilon, s)} m_{k_1} m_{k_2} \cdots m_{k_s}. \tag{11}$$

Note that for $m_k \equiv 1$, as e.g. for the weighted Hermite space and the weighted cosine space, we have

$$n(\varepsilon, \mathrm{APP}_s) = |A(\varepsilon, s)|.$$

For the general case, the set $A(\varepsilon, s)$ is empty for $\varepsilon \geq 1$, and then $n(\varepsilon, \mathrm{APP}_s) = 0$ as we already remarked. Let

$$x(t) = \frac{\log t^{-2}}{\log \omega^{-1}} \quad \text{for all} \;\; t \in (0, \infty). \tag{12}$$

For $s = 1$, it is easy to check that $A(\varepsilon, 1) = \{0, 1, \dots, \lceil (x(\varepsilon)/a_1)^{1/b_1} \rceil - 1\}$ and

$$n(\varepsilon, \mathrm{APP}_1) = m_0 + m_1 + \cdots + m_{\lceil (x(\varepsilon)/a_1)^{1/b_1} \rceil - 1}. \tag{13}$$

For $s \geq 2$, we have

$$\begin{aligned} A(\varepsilon, s) &= \bigcup_{k=0}^{\infty} \left\{ \boldsymbol{k} \in \mathbb{N}_0^{s-1} \times \{k\} \,:\, \sum_{j=1}^{s-1} a_j k_j^{b_j} < x(\varepsilon) - a_s k^{b_s} \right\} \\ &= \bigcup_{k=0}^{\lceil (x(\varepsilon)/a_s)^{1/b_s} \rceil - 1} \left\{ \boldsymbol{k} \in \mathbb{N}_0^{s-1} \times \{k\} \,:\, \sum_{j=1}^{s-1} a_j k_j^{b_j} < x(\varepsilon) - a_s k^{b_s} \right\}. \end{aligned}$$

Since $x(\varepsilon) - a_s k^{b_s} = (\log (\varepsilon\, \omega^{-a_s k^{b_s}/2})^{-2}) / \log \omega^{-1}$, we obtain from (11)

$$n(\varepsilon, \mathrm{APP}_s) = \sum_{k=0}^{\lceil (x(\varepsilon)/a_s)^{1/b_s} \rceil - 1} m_k \, n\left(\varepsilon\, \omega^{-a_s k^{b_s}/2}, \mathrm{APP}_{s-1}\right). \tag{14}$$

For $\varepsilon_1 \leq \varepsilon_2$ we have $n(\varepsilon_2, \mathrm{APP}_s) \leq n(\varepsilon_1, \mathrm{APP}_s)$. Since $\varepsilon \leq \varepsilon\, \omega^{-a_s k^{b_s}/2}$ for all $k \in \mathbb{N}_0$, we conclude that

$$n(\varepsilon, \mathrm{APP}_s) \leq \left(\sum_{k=0}^{\lceil (x(\varepsilon)/a_s)^{1/b_s} \rceil - 1} m_k \right) n(\varepsilon, \mathrm{APP}_{s-1}) \quad \text{for all} \quad s \geq 2. \tag{15}$$

We obtain a lower bound on $n(\varepsilon, \mathrm{APP}_s)$ if we consider only the term $k = 0$ in (14). Then

$$n(\varepsilon, \mathrm{APP}_s) \geq m_0 \, n(\varepsilon, \mathrm{APP}_{s-1}) \quad \text{for all} \quad s \geq 2.$$

For $\varepsilon \geq \omega^{a_s/2}$ we have

$$n(\varepsilon, \mathrm{APP}_s) = m_0 \, n(\varepsilon, \mathrm{APP}_{s-1}) \quad \text{for all} \quad s \geq 2$$

since $\varepsilon\, \omega^{-a_s k^{b_s}/2} \geq 1$ for all positive k and the terms in (14) for $k > 0$ are zero.

For $x(\varepsilon) > a_1$, define

$$j(\varepsilon) = \sup\{\, j \in \mathbb{N} \,:\, x(\varepsilon) > a_j \,\}. \tag{16}$$

Obviously, $j(\varepsilon) \geq 1$. For $\lim_j a_j < \infty$ (here and in the following we use the convention that we write $\lim_j$ instead of $\lim_{j \to \infty}$), we have $j(\varepsilon) = \infty$ for small ε. On the other hand, if $\lim_j a_j = \infty$ we can replace the supremum in the definition of $j(\varepsilon)$ by the maximum and $j(\varepsilon)$ is finite for all ε with $x(\varepsilon) > a_1$. However, $j(\varepsilon)$ tends to infinity as ε tends to zero.

If $j(\varepsilon)$ is finite then

$$n(\varepsilon, \mathrm{APP}_s) = m_0^{s - j(\varepsilon)} \, n(\varepsilon, \mathrm{APP}_{j(\varepsilon)}) \quad \text{for all} \quad s \geq j(\varepsilon).$$

Indeed, for $j \in (j(\varepsilon), s]$ we have $x(\varepsilon) \leq a_j$ and $x(\varepsilon) - a_j k^{b_j} \leq 0$ for all $k \geq 1$. This implies that $\varepsilon\, \omega^{-a_j k^{b_j}/2} \geq 1$ for all $k \geq 1$, and the sum in (14) reduces to one term for

$k=0$. Hence, $n(\varepsilon,\mathrm{APP}_s)=m_0\,n(\varepsilon,\mathrm{APP}_{s-1})=\cdots=m_0^{s-j(\varepsilon)}n(\varepsilon,\mathrm{APP}_{j(\varepsilon)})$, as claimed. Therefore, if $j(\varepsilon)<\infty$ and $m_0=1$ then $n(\varepsilon,\mathrm{APP}_s)$ is independent of s for large s, and

$$\lim_{s\to\infty}\frac{\log\, n(\varepsilon,\mathrm{APP}_s)}{s}=0.$$

Recall that we assume that the sequence $\boldsymbol{m}=\{m_k\}_{k\in\mathbb{N}_0}$ of multiplicities is bounded. That is,

$$m_{\max}=\max_{k\in\mathbb{N}}\ m_k$$

is well defined and $m_{\max}<\infty$. We also set

$$m_{\min}=\min_{k\in\mathbb{N}}\ m_k.$$

Clearly, $m_{\min}\ge 1$.

We are ready to prove the following lemma.

Lemma 1.

Let $x(\varepsilon)$, $j(\varepsilon)$, $m_{\max}$ and $m_{\min}$ be defined as above.

(i) For $\varepsilon\in(0,1)$ we have

$$n(\varepsilon,\mathrm{APP}_s)\ge m_0^s,$$

whereas for $\varepsilon\in(0,1)$ and $x(\varepsilon)\le a_1$ we have

$$n(\varepsilon,\mathrm{APP}_s)=m_0^s.$$

(ii) For $x(\varepsilon)>a_1+a_2+\cdots+a_s$ we have

$$n(\varepsilon,\mathrm{APP}_s)\ge(m_0+m_1)^s.$$

(iii) For $x(\varepsilon)>a_1$ and $\varepsilon\in(0,1)$ we have

$$n(\varepsilon,\mathrm{APP}_s)\le m_0^s\prod_{j=1}^{\min(s,j(\varepsilon))}\left(1+\frac{m_{\max}}{m_0}\left(\left\lceil\left(\frac{x(\varepsilon)}{a_j}\right)^{1/b_j}\right\rceil-1\right)\right).$$

(iv) For $x(\varepsilon)>a_1$, $\varepsilon\in(0,1)$, and arbitrary $\alpha_j\in[0,1]$ we have

$$n(\varepsilon,\mathrm{APP}_s)\ge m_0^s\prod_{j=1}^{\min(s,j(\varepsilon))}\left(1+\frac{m_{\min}}{m_0}\left(\left\lceil\left(\frac{x(\varepsilon)}{a_j}(1-\alpha_j)\prod_{k=j+1}^{s}\alpha_k\right)^{1/b_j}\right\rceil-1\right)\right).$$

In particular, for $\alpha_j=(j-1)/j$ we have

$$n(\varepsilon,\mathrm{APP}_s)\ge m_0^s\prod_{j=1}^{\min(s,j(\varepsilon))}\left(1+\frac{m_{\min}}{m_0}\left(\left\lceil\left(\frac{x(\varepsilon)}{a_j\,s}\right)^{1/b_j}\right\rceil-1\right)\right).$$

Proof. To prove (i), observe that for $\varepsilon \in (0,1)$ the set $A(\varepsilon, s)$ is nonempty since $\boldsymbol{k} = \boldsymbol{0} \in A(\varepsilon, s)$. Therefore (11) yields $n(\varepsilon, \mathrm{APP}_s) \geq m_0^s$. Furthermore, for $x(\varepsilon) \leq a_1$ the set $A(\varepsilon, s) = \{\boldsymbol{0}\}$ and therefore $n(\varepsilon, \mathrm{APP}_s) = m_0^s$, as claimed.

To prove (ii), observe that all $\boldsymbol{k} \in \{0,1\}^s$ belong to the set $A(\varepsilon, s)$. Therefore

$$n(\varepsilon, \mathrm{APP}_s) \geq \sum_{k_1, k_2, \ldots, k_s = 0}^{1} m_{k_1} m_{k_2} \cdots m_{k_s} = (m_0 + m_1)^s,$$

as claimed.

To prove (iii), we first take $s = 1$. Then (13) yields

$$n(\varepsilon, \mathrm{APP}_1) \leq m_0 + m_{\max} \left(\left\lceil \left(\frac{x(\varepsilon)}{a_1} \right)^{1/b_1} \right\rceil - 1 \right) = m_0 \left(1 + \frac{m_{\max}}{m_0} \left(\left\lceil \left(\frac{x(\varepsilon)}{a_1} \right)^{1/b_1} \right\rceil - 1 \right) \right),$$

as needed. For $s \geq 2$ we use (15) and obtain

$$n(\varepsilon, \mathrm{APP}_s) \leq m_0 \left(1 + \frac{m_{\max}}{m_0} \left(\left\lceil \left(\frac{x(\varepsilon)}{a_s} \right)^{1/b_s} \right\rceil - 1 \right) \right) n(\varepsilon, \mathrm{APP}_{s-1}).$$

This implies that

$$n(\varepsilon, \mathrm{APP}_s) \leq m_0^s \prod_{j=1}^{s} \left(1 + \frac{m_{\max}}{m_0} \left(\left\lceil \left(\frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \right\rceil - 1 \right) \right).$$

Note that for $j(\varepsilon) < s$ we have $x(\varepsilon) \leq a_j$ for all $j \in [j(\varepsilon)+1, s]$ and therefore

$$1 + \frac{m_{\max}}{m_0} \left(\left\lceil \left(\frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \right\rceil - 1 \right) = 1.$$

This means that we can restrict the product to j up to $j(\varepsilon)$. This completes the proof of (iii).

To prove (iv), it is enough to prove that

$$n(\varepsilon, \mathrm{APP}_s) \geq m_0^s \prod_{j=1}^{s} \left(1 + \frac{m_{\min}}{m_0} \left(\left\lceil \left(\frac{x(\varepsilon)}{a_j} (1 - \alpha_j) \prod_{k=j+1}^{s} \alpha_k \right)^{1/b_j} \right\rceil - 1 \right) \right)$$

since for $j \in (j(\varepsilon), s]$ we have $x(\varepsilon) \leq a_j$ and the corresponding factors are one.

Take first $s = 1$. Then (13) yields

$$\begin{aligned} n(\varepsilon, \mathrm{APP}_1) &\geq m_0 \left(1 + \frac{m_{\min}}{m_0} \left(\left\lceil \left(\frac{x(\varepsilon)}{a_1} \right)^{1/b_1} \right\rceil - 1 \right) \right) \\ &\geq m_0 \left(1 + \frac{m_{\min}}{m_0} \left(\left\lceil \left(\frac{x(\varepsilon)}{a_1} (1 - \alpha_1) \right)^{1/b_1} \right\rceil - 1 \right) \right), \end{aligned}$$

as needed. For $s \geq 2$, note that $x(\varepsilon) - a_s k^{b_s} > \alpha_s x(\varepsilon)$ for all $k \in \mathbb{N}$ for which

$$k \leq \left\lceil \left(\frac{x(\varepsilon)(1 - \alpha_s)}{a_s} \right)^{1/b_s} \right\rceil - 1.$$

For such k we have $\varepsilon\omega^{-a_s k^{b_s}/2} < \varepsilon^{\alpha_s}$ and therefore from (14) we obtain

$$
\begin{aligned}
n(\varepsilon, \mathrm{APP}_s) &\geq \sum_{k=0}^{\lceil (x(\varepsilon)(1-\alpha_s)/a_s)^{1/b_s}\rceil - 1} m_k\, n(\varepsilon^{\alpha_s}, \mathrm{APP}_{s-1}) \\
&\geq m_0 \left(1 + \frac{m_{\min}}{m_0}\left(\left\lceil \left(\frac{x(\varepsilon)(1-\alpha_s)}{a_s}\right)^{1/b_s} \right\rceil - 1\right)\right) n(\varepsilon^{\alpha_s}, \mathrm{APP}_{s-1}).
\end{aligned}
$$

Since $x(\varepsilon^{\alpha_s}) = \alpha_s x(\varepsilon)$, the proof is completed by applying induction on s. For $\alpha_j = (j-1)/j$ we have $(1-\alpha_j)\prod_{k=j+1}^{s}\alpha_k = 1/s$, which completes the proof. $\square$

4 Exponential Convergence

As in [2, 3, 8, 10], by exponential convergence (EXP) we mean that the nth minimal errors $e(n, \mathrm{APP}_s)$ are bounded by

$$e(n, \mathrm{APP}_s) \leq C_s\, q^{(n/M_s)^{p_s}} \qquad \text{for all } \ n \in \mathbb{N},$$

for some positive C_s, M_s and p_s with $q \in (0,1)$. The supremum of p_s for which the last bound holds is denoted by p_s^* and is called the exponent of EXP for the s-variate case. We also have the concept of uniform exponential convergence (UEXP) if we can take $p_s = p > 0$ for all $s \in \mathbb{N}$. Then the supremum of such p is denoted by p^* and is called the exponent of UEXP.

We want to verify when EXP and UEXP hold for the approximation problem $\mathrm{APP} = \{\mathrm{APP}_s\}_{s\in\mathbb{N}}$ in terms of the varying parameters $\boldsymbol{a} = \{a_s\}_{s\in\mathbb{N}}$ and $\boldsymbol{b} = \{b_s\}_{s\in\mathbb{N}}$, which define the domain spaces $H_{s,\boldsymbol{a},\boldsymbol{b}}$ of APP_s and satisfy (3).

Theorem 1.

Consider the approximation problem $\mathrm{APP} = \{\mathrm{APP}_s\}_{s\in\mathbb{N}}$ *with the embedding operators* $\mathrm{APP}_s : H_{s,\boldsymbol{a},\boldsymbol{b}} \to H_s$. *Then*

(i) EXP holds for arbitrary $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *with the exponent*

$$p_s^* = \frac{1}{B_s} \qquad \textit{and} \qquad B_s := \sum_{j=1}^{s}\frac{1}{b_j}.$$

(ii) UEXP holds iff $\boldsymbol{a}$ *is arbitrary and* $\boldsymbol{b}$ *is such that*

$$B := \sum_{j=1}^{\infty}\frac{1}{b_j} < \infty.$$

If $B < \infty$ *then the exponent of UEXP is* $p^* = 1/B$.

Proof. From (iii) and (iv) of Lemma 1 with a fixed s we conclude that there are positive numbers $c_1(s)$ and $c_2(s)$ such that

$$c_1(s)[x(\varepsilon)]^{B_s} \leq n(\varepsilon, \mathrm{APP}_s) \leq c_2(s)\,[x(\varepsilon)]^{B_s} \qquad \text{for all } \ \varepsilon \in (0,1).$$

Clearly, $x(\varepsilon) = \Theta(\log \varepsilon^{-1})$. Therefore

$$n(\varepsilon, \mathrm{APP}_s) = \Theta\left([\log\ \varepsilon^{-1}]^{B_s}\right).$$

From this it follows that we can find positive $c_j(s)$ for $j = 3, 4, 5, 6$ such that

$$c_3(s)\,\mathrm{e}^{-(n/c_4(s))^{1/B_s}} \le e(n, \mathrm{APP}_s) \le c_5(s)\,\mathrm{e}^{-(n/c_6(s))^{1/B_s}} \quad \text{for all} \;\; n \in \mathbb{N},$$

where $\mathrm{e} = \exp(1)$. This proves EXP with $p_s^* = 1/B_s$, as claimed.

We now turn to UEXP. Suppose that UEXP holds. Then $e(n, \mathrm{APP}_s) \le C_s\, q^{(n/M_s)^p}$. This implies that

$$n(\varepsilon, \mathrm{APP}_s) = \Theta\left([\log\ \varepsilon^{-1}]^{1/p}\right).$$

Thus, $B_s \le 1/p$ for all $s \in \mathbb{N}$. Therefore $B \le 1/p < \infty$ and $p^* \le 1/B$. On the other hand, if $B < \infty$ then we can set $p_s = 1/B$ and obtain UEXP. Hence, $p^* \ge 1/B$, and therefore $p^* = 1/B$. This completes the proof. $\square$

We stress that EXP and UEXP hold for arbitrary sequences $\boldsymbol{m}$ of multiplicity and the only condition is on $\boldsymbol{b}$ for UEXP. This is true since the concepts of EXP and UEXP do not specify how C_s and M_s depend on s. In fact, in general, it is easy to see from Lemma 1 that $c_1(s)$ and $c_2(s)$, as well as the other $c_j(s)$, depend exponentially on s. It is especially clear if the multiplicity $m_0 \ge 2$. If we wish to control the dependence on s and to control the exponential dependence on s then we need to study tractability which is the subject of the next section.

5 Tractability

Tractability studies how the information complexity depends on both ε^{-1} and s. The key point is to characterize when this dependence is not exponential in $(s^{t_1}, \varepsilon^{-t_2})$ or in $(s^{t_1}, (1 + \log \varepsilon^{-1})^{t_2})$ for some positive t_1 and t_2, and when this dependence is polynomial in (s, ε^{-1}) or in $(s, 1 + \log \varepsilon^{-1})$. For $t_1 = t_2 = 1$, the survey of tractability results for general multivariate problems and for the pair (s, ε^{-1}) can be found in [12, 13, 14], and for more specific multivariate problems and the pair $(s, 1 + \log \varepsilon^{-1})$ in [2, 3, 8, 10].

We will cover a number of tractability notions and verify when they hold for the approximation problem $\mathrm{APP} = \{\mathrm{APP}_s\}_{s\in\mathbb{N}}$ in terms of the parameters $\boldsymbol{a}, \boldsymbol{b}$, $\boldsymbol{m}$ and ω. We will analyze the tractability notions starting from the weakest notions and continuing to the strongest notions. A table which gives an overview of the obtained tractability results is presented in Section 6.

5.1 Standard Notions of Tractability

By the standard notions of tractability we mean tractability notions with respect to the pair (s, ε^{-1}).

- **(t_1, t_2)-Weak Tractability**

  As in [18], we say that APP is (t_1, t_2)-weakly tractable (shortly (t_1, t_2)-WT) for positive t_1 and t_2 iff

  $$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} = 0.$$

This means that $n(\varepsilon, \mathrm{APP}_s)$ is *not* exponential in $s^{\,t_1}$ and ε^{-t_2} but it may be exponential in s^{τ_1} or $\varepsilon^{-\tau_2}$ for positive $\tau_1 < t_1$ or $\tau_2 < t_2$. In particular, if $t_1 > 1$ we may have the exponential dependence on s which is called the curse of dimensionality.

Theorem 2.

APP *is* (t_1, t_2)-WT *for the parameters* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ *and* ω *iff* $t_1 > 1$ *or* $m_0 = 1$.

Proof. Suppose that APP is (t_1, t_2)-WT for the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ and ω. Then for fixed $\varepsilon \in (0, 1)$ we obtain from (i) of Lemma 1 that

$$0 = \lim_{s\to\infty} \frac{n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} \geq \lim_{s\to\infty} \frac{s \log m_0}{s^{t_1} + \varepsilon^{-t_2}} = \lim_{s\to\infty} s^{1-t_1} \log m_0$$

and hence we must have $t_1 > 1$ or $m_0 = 1$.

Suppose now that $t_1 > 1$. We first show that the hardest case of APP is for constant $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e., $a_j \equiv a_1$ and $b_j \equiv b_1$. Indeed, the eigenvalues of W_s, which define $n(\varepsilon, \mathrm{APP}_s)$, are $\omega^{\sum_{j=1}^s a_j [k(n_j)]^{b_j}}$. Clearly,

$$\sum_{j=1}^{s} a_j [k(n_j)]^{b_j} \geq \sum_{j=1}^{s} a_1 [k(n_j)]^{b_0},$$

where $b_0 = \inf_j b_j$. Due to (3), we have $b_0 > 0$. Therefore

$$\omega^{\sum_{j=1}^s a_j [k(n_j)]^{b_j}} \leq \omega^{\sum_{j=1}^s a_1 [k(n_j)]^{b_0}},$$

and $n(\varepsilon, \mathrm{APP}_s)$ is maximized for $a_j \equiv a_1$ and $b_j \equiv b_0$ (and just now $b_0 = b_1$).

Hence, it is enough to show (t_1, t_2)-WT for constant $\boldsymbol{a}$ and $\boldsymbol{b}$. From (iii) of Lemma 1 we have

$$\log n(\varepsilon, \mathrm{APP}_s) \leq s \left(\log\, m_0 \;+\; \log \left(1 + \frac{m_{\max} 2^{1/b_1}}{m_0 (a_1 \log\, \omega^{-1})^{1/b_1}} \, [\log\, \varepsilon^{-1}]^{1/b_1} \right) \right).$$

This shows that for small ε we have

$$\log\, n(\varepsilon, \mathrm{APP}_s) = \mathcal{O}(s\, \log\, \log\, \varepsilon^{-1}) \tag{17}$$

with the factor in the big $\mathcal{O}$ notation independent of s and ε^{-1}. Hence,

$$\frac{\log\, n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} = \mathcal{O}\left(\frac{s\, \log\, \log\, \varepsilon^{-1}}{s^{t_1} + \varepsilon^{-t_2}} \right).$$

Let $y = \max(s^{t_1}, \varepsilon^{-t_2})$. Then $\varepsilon^{-1} \leq y^{1/t_2}$, $s \leq y^{1/t_1}$ and

$$\frac{s\, \log\, \log\, \varepsilon^{-1}}{s^{t_1} + \varepsilon^{-t_2}} \leq \frac{y^{1/t_1}\, \log\, \log\, y^{1/t_2}}{y} = \frac{\log\, \log\, y^{1/t_2}}{y^{1-1/t_1}}$$

and it goes to zero as $s + \varepsilon^{-1}$, or equivalently y, approaches infinity since $t_1 > 1$ and $t_2 > 0$. This proves (t_1, t_2)-WT.

Finally, suppose that $m_0 = 1$. Then the second largest eigenvalue for all s is $\lambda_{1,2} = \omega^{a_1}$, which is smaller than the largest eigenvalue $\lambda_{s,1} = 1$. As above it suffices to consider APP for constant $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e. $a_j \equiv a_1$ and $b_j \equiv b_1$. In this case, we can use an estimate for the information complexity which has been shown in [15, p. 611], and which states

$$n(\varepsilon, \mathrm{APP}_s) \leq \frac{s!}{(s-a_s(\varepsilon))!} \prod_{j=1}^{a_s(\varepsilon)} n(\varepsilon^{1/j}, \mathrm{APP}_1),$$

where

$$a_s(\varepsilon) = \min \left\{ s, \left\lfloor 2 \frac{\log \varepsilon^{-1}}{\log \omega^{-a_1}} \right\rfloor - 1 \right\}.$$

Then we have

$$\log n(\varepsilon, \mathrm{APP}_s) \leq \log \frac{s!}{(s-a_s(\varepsilon))!} + \sum_{j=1}^{a_s(\varepsilon)} \log n(\varepsilon^{1/j}, \mathrm{APP}_1). \tag{18}$$

From (13) with the assumption $m_0 = 1$ we obtain

$$n(\varepsilon^{1/j}, \mathrm{APP}_1) \leq 1 + m_{\max} \left(\frac{x(\varepsilon^{1/j})}{a_1} \right)^{1/b_1} \leq 2\, m_{\max} \max \left(1, \left(\frac{x(\varepsilon^{1/j})}{a_1} \right)^{1/b_1} \right).$$

Note that

$$\left(\frac{x(\varepsilon^{1/j})}{a_1} \right)^{1/b_1} = \left(\frac{2}{a_1\, j} \frac{\log\, \varepsilon^{-1}}{\log\, \omega^{-1}} \right)^{1/b_1} \leq \left(\frac{2}{a_1} \frac{\log\, \varepsilon^{-1}}{\log\, \omega^{-1}} \right)^{1/b_1}.$$

Assume that $\varepsilon \leq \omega^{a_1/2}$. Then the last right hand side is at least one and therefore

$$\begin{aligned} \log n(\varepsilon^{1/j}, \mathrm{APP}_1) &\leq \log \left(2\, m_{\max} \left(\frac{2}{a_1} \right)^{1/b_1} \right) + \frac{1}{b_1} \log \left(\frac{\log \varepsilon^{-1}}{\log \omega^{-1}} \right) \\ &= C_1 + C_2 \log\log \varepsilon^{-1}, \end{aligned}$$

where $C_1 = \log(2\, m_{\max}(\frac{2}{a_1})^{1/b_1}) - \frac{1}{b_1} \log\log \omega^{-1}$ and $C_2 = \frac{1}{b_1}$. Hence we obtain

$$\begin{aligned} \sum_{j=1}^{a_s(\varepsilon)} \log n(\varepsilon^{1/j}, \mathrm{APP}_1) &\leq a_s(\varepsilon)(C_1 + C_2 \log\log \varepsilon^{-1}) \\ &\leq C_3 \log \varepsilon^{-1} \log\log \varepsilon^{-1} \end{aligned} \tag{19}$$

with a suitable $C_3 > 0$. Hence we have

$$\limsup_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} \leq \limsup_{s+\varepsilon^{-1}\to\infty} \frac{\log \frac{s!}{(s-a_s(\varepsilon))!}}{s^{t_1} + \varepsilon^{-t_2}}.$$

Since

$$\frac{s!}{(s-a_s(\varepsilon))!} = (s - a_s(\varepsilon) + 1)(s - a_s(\varepsilon) + 2) \cdots s \leq s^{a_s(\varepsilon)}$$

we have

$$\log \frac{s!}{(s-a_s(\varepsilon))!} \le a_s(\varepsilon)\, \log\, s = \mathcal{O}(\log\, \varepsilon^{-1}\, \log\, s). \tag{20}$$

As before, let $y = \max(s^{t_1}, \varepsilon^{-t_2})$. Since $t_1 > 0$ and $t_2 > 0$ we have $\varepsilon^{-1} \le y^{1/t_2}$, $s \le y^{1/t_1}$ and

$$\frac{\log\, \varepsilon^{-1}\, \log\, s}{s^{t_1} + \varepsilon^{-t_2}} \le \frac{[\log\, y]^2}{t_1 t_2\, y}$$

goes to zero as $s^{t_1} + \varepsilon^{-t_2}$, or equivalently y, approaches infinity. Hence

$$\limsup_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} = \lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} = 0.$$

□

- **Weak and Uniform Weak Tractability**

  Weak tractability (WT) corresponds to (t_1, t_2)-WT for $t_1 = t_2 = 1$. Uniform weak tractability (UWT) holds iff we have (t_1, t_2)-WT for all $t_1, t_2 \in (0, 1]$.

  Theorem 3.

  APP *is* WT *iff* APP *is* UWT *iff* $m_0 = 1$.

  Proof. Since UWT implies WT, it is enough to show that WT implies $m_0 = 1$, and that $m_0 = 1$ implies UWT. Suppose then that APP is WT. From the previous proof we conclude that $m_0 = 1$. On the other hand, if $m_0 = 1$ then APP is not only UWT but it is quasi-polynomially tractable which is a stronger notion than UWT. This will be shown in a moment. □

- **Quasi-Polynomial Tractability**

  APP is quasi-polynomially tractable (QPT) iff there are positive numbers C and t such that

$$n(\varepsilon, \mathrm{APP}_s) \le C\, \exp\big(t(1+\log\, s)(1+\log\, \varepsilon^{-1})\big) \qquad \text{for all } \; s \in \mathbb{N},\; \varepsilon \in (0,1).$$

  The infimum of t satisfying the bound above is denoted by t^*, and is called the exponent of QPT. Clearly, QPT implies UWT.

  Theorem 4.

  APP *is* QPT *iff* $m_0 = 1$.

  If $m_0 = 1$ *then the exponent of* QPT *is* $t^* \le \frac{2}{a_1\, \log\, \omega^{-1}}$ *and the last bound becomes an equality for constant* $\boldsymbol{a}$ *and* $\boldsymbol{b}$.

  Proof. Suppose that APP is QPT. Then APP is UWT and $m_0 = 1$.

  We now show that $m_0 = 1$ implies QPT and $t^* \le 2/(a_1\, \log\, \omega^{-1})$. As before, it is enough to prove it for constant $\boldsymbol{a}$ and $\boldsymbol{b}$. In this case $H_{s,\boldsymbol{a},\boldsymbol{b}}$ is the tensor product of s copies of H_{a_1,b_1}. Then the eigenvalues $\{\lambda_{s,k}\}_{k\in\mathbb{N}}$ of W_s are products of the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ of W_1, i.e., $\{\lambda_{s,k}\}_{k\in\mathbb{N}} = \{\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_s}\}_{k_1,k_2,\dots,k_s\in\mathbb{N}}$ with the

ordered (distinct) eigenvalues $\lambda_k = \omega^{a_1(k-1)^{b_1}}$ for $k \in \mathbb{N}$. It is proved in [7] that APP is QPT iff $\lambda_2 < \lambda_1$ and $\mathrm{decay}_\lambda := \sup\{r \,:\, \lim_k k^r \lambda_k = 0\} > 0$. If so then

$$t^* = \max\left(\frac{2}{\mathrm{decay}_\lambda}, \frac{2}{\log\frac{\lambda_1}{\lambda_2}}\right).$$

In our case, $\lambda_1 = 1$ and $\lambda_2 = \omega^{a_1}$ so that the assumption $\lambda_2 < \lambda_1$ holds. Furthermore $\lim_k k^r\, \omega^{a_1(k-1)^{b_1}} = 0$ for all $r > 0$, so that $\mathrm{decay}_\lambda = \infty$. Hence, $t^* = 2/(a_1 \log \omega^{-1})$, as claimed. $\square$

- **Polynomial and Strong Polynomial Tractability**

APP is polynomially tractable (PT) iff there are positive C, p and $q \geq 0$ such that

$$n(\varepsilon, \mathrm{APP}_s) \leq C\, s^{\,q}\, \varepsilon^{-p} \quad \text{for all} \;\; s \in \mathbb{N},\;\varepsilon \in (0,1).$$

APP is strongly polynomially tractable (SPT) iff the last bound holds for $q = 0$. Then the infimum of p in the bound above is denoted by p^*, and is called the exponent of SPT. For simplicity, we assume that

$$\alpha := \lim_{j\to\infty} \frac{a_j}{\log\, j}$$

exists.

Theorem 5.

APP *is* SPT *iff* APP *is* PT *iff*

$$m_0 = 1 \quad \textit{and} \quad \alpha > 0.$$

If this is the case then the exponent of SPT *is* $p^* = \frac{2}{\alpha \log \omega^{-1}}$.

Proof. We use [12, Theorem 5.2] which states necessary and sufficient conditions on PT and SPT in terms of the eigenvalues $\{\lambda_{s,k}\}_{k\in\mathbb{N}}$ of W_s. For our problem we have $\lambda_{s,1} = 1$. Namely, APP is PT iff there are numbers $q \geq 0$ and $\tau > 0$ such that

$$\sup_{s\in\mathbb{N}} \left(\sum_{k=1}^{\infty} \lambda_{s,k}^{\tau}\right)^{1/\tau} s^{\,-q} < \infty, \tag{21}$$

and it is SPT iff the last inequality holds with $q = 0$. Then the exponent p^* of SPT is the infimum of 2τ for τ satisfying (21) with $q = 0$.

In our case,

$$\sum_{k=1}^{\infty} \lambda_{s,k}^{\tau} = \sum_{\boldsymbol{n}\in\mathbb{N}_0^s} \prod_{j=1}^{s} \omega^{\tau\, a_j [k(n_j)]^{b_j}} = \prod_{j=1}^{s} \left(m_0 + \sum_{k=1}^{\infty} m_k \omega^{\tau\, a_j\, k^{b_j}}\right).$$

Let $b_0 = \inf_j b_j$ and $C_\tau = m_{\max} \sum_{k=1}^{\infty} \omega^{\tau a_1 (k^{b_0}-1)}$. Then $b_0 > 0$ and $C_\tau < \infty$ for all $\tau > 0$. Furthermore,

$$m_1\, \omega^{\tau\, a_j} \leq \sum_{k=1}^{\infty} m_k \omega^{\tau\, a_j\, k^{b_j}} \leq m_{\max}\, \omega^{\tau\, a_j} \sum_{k=1}^{\infty} \omega^{\tau\, a_j (k^{b_j}-1)} \leq C_\tau\, \omega^{\tau\, a_j}.$$

Therefore

$$\prod_{j=1}^{s}(m_0+m_1\omega^{\tau\,a_j})\le\prod_{j=1}^{s}\left(m_0+\sum_{k=1}^{\infty}m_k\omega^{\tau\,a_j\,k^{b_j}}\right)\le\prod_{j=1}^{s}(m_0+C_\tau\,\omega^{\tau\,a_j})\,. \tag{22}$$

Let $\omega^{\tau\,a_j}=(j+1)^{-x_j}$. That is,

$$x_j=\frac{a_j}{\log(j+1)}\,\tau\,\log\omega^{-1}\quad\text{and}\quad\lim_{j\to\infty}x_j=\alpha\,\tau\,\log\omega^{-1}.$$

Then

$$\prod_{j=1}^{s}\left(m_0+\frac{m_1}{(j+1)^{x_j}}\right)\le\prod_{j=1}^{s}\left(m_0+\sum_{k=1}^{\infty}m_k\omega^{\tau\,a_j\,k^{b_j}}\right)\le\prod_{j=1}^{s}\left(m_0+\frac{C_\tau}{(j+1)^{x_j}}\right).$$

Hence, (21) holds iff $m_0=1$ and $\lim_j x_j\ge 1$. Indeed, $m_0=1$ is clear because otherwise we have an exponential dependence on s. For $m_0=1$, let $\beta\in\{m_1,C_\tau\}$. Then

$$\prod_{j=1}^{s}\left(m_0+\frac{\beta}{(j+1)^{x_j}}\right)=\exp\left(\sum_{j=1}^{s}\log(1+\beta\,(j+1)^{-x_j})\right).$$

Furthermore,

$$\sum_{j=1}^{s}\log(1+\beta\,(j+1)^{-x_j})=\Theta\left(\sum_{j=1}^{s}(j+1)^{-x_j}\right),$$

with the factors in the big Θ notation independent of s and j.

Suppose that $\alpha=0$. Then $\lim_j x_j=0$ for all τ. This means for all $\delta\in(0,1)$ there is an integer $j(\delta,\tau)$ such that $x_j\le\delta$ for all $j\ge j(\delta,\tau)$, and $\sum_{j=1}^{s}(j+1)^{-x_j}=\Theta(s^{1-\delta})$. Hence

$$\prod_{j=1}^{s}\left(m_0+\frac{\beta}{(j+1)^{x_j}}\right)\quad\text{as well as}\quad\left(\sum_{k=1}^{\infty}\lambda_{s,k}^{\tau}\right)^{1/\tau}$$

is exponential in $s^{1-\delta}$. This means that (21) does not hold for any positive τ and non-negative q. Hence, we do not have PT.

Suppose now that $\alpha>0$. Then $\lim_j x_j>1$ for $\tau>(\alpha\,\log\omega^{-1})^{-1}$. This implies that

$$\sum_{j=1}^{s}(j+1)^{-x_j}\quad\text{as well as}\quad\left(\sum_{k=1}^{\infty}\lambda_{s,k}^{\tau}\right)^{1/\tau}$$

is uniformly bounded in s. Hence, (21) holds for $q=0$ and we have SPT with the exponent $p^*\le 2/(\alpha\,\log\omega^{-1})$. For $\tau<(\alpha\,\log\omega^{-1})^{-1}$, the series $\sum_{j=1}^{s}(j+1)^{-x_j}$ is of order at least $\log s$ and (21) may hold only for $q>0$. This contradicts SPT. Hence $p^*\ge 2/(\alpha\,\log\omega^{-1})$, which completes the proof. □

5.2 New Notions of Tractability

We now turn to new notions of tractability which correspond to the standard notions of tractability for the pair $(s, 1 + \log\,\varepsilon^{-1})$ instead of the pair (s, ε^{-1}). To distinguish between the standard and new notions of tractability, we add the prefix EC (exponential convergence) when we consider the new notions. As before, we study the new notions of tractability for the approximation problem $\mathrm{APP} = \{\mathrm{APP}_s\}_{s\in\mathbb{N}}$ for general parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ and ω.

- **EC-(t_1, t_2)-Weak Tractability**

  We say that APP is EC-(t_1, t_2)-WT iff

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log\, n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + [\log\,\varepsilon^{-1}]^{t_2}} = 0.$$

  Obviously, EC-(t_1, t_2)-WT implies (t_1, t_2)-WT. For $t_1 = 1$ and $t_2 > 1$, this notion was introduced and studied in [15].

Theorem 6.

APP *is* EC-(t_1, t_2)-WT *for the parameters* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ *and* ω *iff* $t_1 > 1$*, or* $t_2 > 1$ *and* $m_0 = 1$*.*

Proof. Suppose that APP is EC-(t_1, t_2)-WT for the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ and ω. Then for fixed $\varepsilon \in (0, 1)$ we obtain from (i) of Lemma 1 that

$$0 = \lim_{s\to\infty} \frac{\log n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + [\log\varepsilon^{-1}]^{t_2}} \geq \lim_{s\to\infty} \frac{s \log m_0}{s^{t_1} + [\log\varepsilon^{-1}]^{t_2}} = \lim_{s\to\infty} s^{1-t_1} \log m_0.$$

Hence, we conclude that $t_1 > 1$ or that $m_0 = 1$. For $t_1 \leq 1$ and $m_0 = 1$, it remains to show that $t_2 > 1$. As in [15, p. 609] we find that for $\varepsilon_s \in (\lambda^{(\lfloor s/2\rfloor+1)/2}, \lambda^{\lfloor s/2\rfloor/2}) =: L_s$, where $\lambda := \omega^{a_1}$, we have $n(\varepsilon_s, \mathrm{APP}_s) \geq 2^{\lfloor s/2\rfloor}$. Then

$$0 = \lim_{\substack{s\to\infty\\ \varepsilon_s\in L_s}} \frac{\log n(\varepsilon_s, \mathrm{APP}_s)}{s^{t_1} + [\log\varepsilon_s^{-1}]^{t_2}} \geq \lim_{s\to\infty} \frac{\lfloor s/2\rfloor \log 2}{s^{t_1} + \left(\frac{\lfloor s/2\rfloor+1}{2}\log\lambda^{-1}\right)^{t_2}}.$$

This can only hold if $t_2 > 1$.

Suppose now that $t_1 > 1$. From (17) we have for small ε,

$$\frac{\log\, n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + [\log\,\varepsilon^{-1}]^{t_2}} = \mathcal{O}\left(\frac{s\,\log\,\log\,\varepsilon^{-1}}{s^{t_1} + [\log\,\varepsilon^{-1}]^{t_2}}\right).$$

Let $y = \max(s^{t_1}, [\log\,\varepsilon^{-1}]^{t_2})$. Then

$$\frac{s\,\log\,\log\,\varepsilon^{-1}}{s^{t_1} + [\log\,\varepsilon^{-1}]^{t_2}} \leq \frac{y^{1/t_1}\,\log\,y}{t_2\,y}.$$

Clearly, this goes to zero as $s + \varepsilon^{-1}$ approaches infinity since $t_1 > 1$ and $t_2 > 0$. Hence, we have EC-(t_1, t_2)-WT, as claimed.

Suppose now that $t_2 > 1$ and $m_0 = 1$. From the proof of Theorem 2, (18), (19) and (20) we obtain

$$\log n(\varepsilon, \mathrm{APP}_s) \le C \log \varepsilon^{-1} \log s + C_3 \log \varepsilon^{-1} \log \log \varepsilon^{-1}$$

with suitable constants $C, C_3 > 0$. Since $t_1 > 0$ and $t_2 > 1$ and using the same argument for $y = \max(s^{t_1}, [\log \varepsilon^{-1}]^{t_2})$ as above, it follows that

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + [\log \varepsilon^{-1}]^{t_2}} = 0.$$

Hence, we have EC-(t_1, t_2)-WT. This completes the proof. □

- **EC-Weak and EC-Uniform Weak Tractability**

EC-weak tractability (EC-WT) corresponds to EC-$(1,1)$-WT. EC-uniform weak tractability (EC-UWT) means that EC-(t_1, t_2)-WT holds for all $t_1, t_2 \in (0,1]$. Clearly, EC-WT implies WT, and EC-UWT implies UWT.

Theorem 7.

- APP *is* EC-WT *iff* $m_0 = 1$ *and* $\lim_{j\to\infty} a_j = \infty$,
- APP *is* EC-UWT *iff* $m_0 = 1$ *and* $\lim_{j\to\infty} \frac{\log a_j}{\log j} = \infty$.

Proof. We first assume that EC-WT or EC-UWT holds. Since EC-WT implies WT and EC-UWT implies UWT, Theorem 3 implies that $m_0 = 1$. For $t_1, t_2 \in (0,1]$, let

$$z_{s,t_1,t_2} = \frac{\log\, n(\varepsilon, \mathrm{APP}_s)}{s^{t_1} + [\log\, \varepsilon^{-1}]^{t_2}}.$$

Let $\delta > 0$ and take $x(\varepsilon) = (1+\delta)(a_1 + \cdots + a_s)$. Due to the definition (12) of $x(\varepsilon)$ this means that

$$\log\, \varepsilon^{-1} = \frac{\log\, \omega^{-1}}{2}\,(1+\delta)\,(a_1 + a_2 + \cdots + a_s).$$

From (ii) of Lemma 1 we have

$$z_{s,t_1,t_2} \ge \frac{s\, \log(1+m_1)}{s^{t_1} + [\log\, \varepsilon^{-1}]^{t_2}} \ge \frac{\log\, 2}{s^{t_1-1} + [\frac{1}{2}(1+\delta)\, \log\, \omega^{-1}]^{t_2} y_s},$$

where

$$y_s = \frac{(a_1 + a_2 + \cdots + a_s)^{t_2}}{s} \le \frac{(s\, a_s)^{t_2}}{s} = \frac{a_s^{t_2}}{s^{1-t_2}}.$$

Then $\lim_s z_{s,t_1,t_2} = 0$ implies that $\lim_s y_s = \infty$, which in turn implies that

$$\lim_{s\to\infty} \frac{a_s^{t_2}}{s^{1-t_2}} = \lim_{s\to\infty} \frac{a_s}{s^{t_2^{-1}-1}} = \infty. \tag{23}$$

If we have EC-WT then $\lim_s z_{s,1,1} = 0$ and $\lim_j a_j = \infty$, as claimed. If we have EC-UWT then, in particular, $\lim_s z_{s,1,t_2} = 0$ for all positive t_2. Then (23) yields there is a number $s^* = s^*(t_2)$ such that

$$a_s \ge s^{t_2^{-1}-1} \quad \text{for all} \quad s \ge s^*(t_2),$$

or equivalently

$$\frac{\log\, a_s}{\log\, s} \ge \frac{1-t_2}{t_2} \qquad \text{for all} \ \ s \ge s^*(t_2).$$

Since t_2 can be arbitrarily close to zero this implies that

$$\lim_{s\to\infty} \frac{\log\, a_s}{\log\, s} = \infty,$$

as claimed.

We now prove that $m_0 = 1$ and $\lim_j a_j = \infty$ imply EC-WT. For any positive η we compute the η powers of the eigenvalues $\lambda_{s,k}$ of the operator W_s. We have

$$n\lambda_{s,n}^{\eta} \le \sum_{k=1}^{\infty} \lambda_{s,k}^{\eta} = \prod_{j=1}^{s} \left(1 + \sum_{k=1}^{\infty} m_k\, \omega^{\eta\, a_j\, k^{b_j}}\right) \le \prod_{j=1}^{s} \left(1 + C_\eta\, \omega^{\eta\, a_j}\right),$$

due to (22). Hence,

$$\lambda_{s,n} \le \frac{\prod_{j=1}^{s} \left(1 + C_\eta\, \omega^{\eta\, a_j}\right)^{1/\eta}}{n^{1/\eta}}.$$

Then (10) yields

$$n(\varepsilon, \mathrm{APP}_s) \le \frac{\prod_{j=1}^{s} \left(1 + C_\eta\, \omega^{\eta\, a_j}\right)}{\varepsilon^{2\eta}}.$$

Since $\log(1+x) \le x$ for positive x, we conclude

$$\log\, n(\varepsilon, \mathrm{APP}_s) \le 2\eta\, \log\, \varepsilon^{-1} \ + C_\eta \sum_{j=1}^{s} \omega^{\eta\, a_j}.$$

Observe that $\lim_j a_j = \infty$ implies $\lim_j \omega^{\eta\, a_j} = 0$ and $\lim_s \sum_{j=1}^{s} \omega^{\eta\, a_j}/s = 0$. Therefore

$$\limsup_{s+\varepsilon^{-1}\to\infty} \frac{\log\, n(\varepsilon, \mathrm{APP}_s)}{s + \log\, \varepsilon^{-1}} \le 2\eta.$$

Since η can be arbitrarily small this proves that

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log\, n(\varepsilon, \mathrm{APP}_s)}{s + \log\, \varepsilon^{-1}} = 0.$$

Hence EC-WT holds.

Finally, we prove that $m_0 = 1$ and $\lim_s (\log\, a_s)/\log\, s = \infty$ imply EC-UWT. For $x(\varepsilon) > a_1$, (iii) of Lemma 1 yields

$$\log\, n(\varepsilon, \mathrm{APP}_s) \le \sum_{j=1}^{j(\varepsilon)} \log \left(1 + m_{\max} \left(\frac{x(\varepsilon)}{a_j}\right)^{1/b_j}\right).$$

Let $b_0 = \inf_j b_j > 0$ and

$$\alpha = m_{\max} \left(\frac{2}{a_1\, \log\, \omega^{-1}}\right)^{1/b_0}.$$

From the definition (12) of $x(\varepsilon)$ we then have

$$m_{\max}\left(\frac{x(\varepsilon)}{a_j}\right)^{1/b_j}\le \alpha[\log\,\varepsilon^{-1}]^{1/b_0},$$

and

$$\log\,n(\varepsilon,\mathrm{APP}_s)\le j(\varepsilon)\,\log\left(1+\alpha[\log\,\varepsilon^{-1}]^{1/b_0}\right)=\mathcal{O}\left(j(\varepsilon)\,\log\,\log\,\varepsilon^{-1}\right).$$

We now estimate $j(\varepsilon)$ using the assumption that $\lim_j(\log\,a_j)/\log\,j=\infty$. We know that for all positive τ there is a number j_τ such that

$$a_j\ge j^\tau\quad\text{for all}\quad j\ge j_\tau.$$

This implies that

$$j(\varepsilon)\le\max\left(j_\tau-1,x(\varepsilon)^{1/\tau}\right)=\mathcal{O}\left([\log\,\varepsilon^{-1}]^{1/\tau}\right).$$

Therefore

$$\log\,n(\varepsilon,\mathrm{APP}_s)=\mathcal{O}\left([\log\,\varepsilon^{-1}]^{1/\tau}\,\log\,\log\,\varepsilon^{-1}\right).$$

We stress that the factors in the big $\mathcal{O}$ notation do not depend on s. Then for any positive $t_1,t_2\in(0,1]$ we take $\tau>1/t_2$, and conclude

$$\lim_{s+\varepsilon^{-1}\to\infty}\frac{\log\,n(\varepsilon,\mathrm{APP}_s)}{s^{\,t_1}+[\log\,\varepsilon^{-1}]^{t_2}}=0.$$

This proves EC-UWT, and completes the proof. $\square$

If we compare Theorems 3 and 7, we see that the assumption $m_0=1$ is always needed. However, WT holds for all $\boldsymbol{a}=\{a_j\}_{j\in\mathbb{N}}$, whereas EC-WT requires that $\lim_j a_j=\infty$. Similarly, UWT holds for all $\boldsymbol{a}=\{a_j\}_{j\in\mathbb{N}}$, whereas EC-UWT requires that $\lim_j(\log\,a_j)/\log\,j=\infty$. Hence, a_j's may go to infinity arbitrarily slowly for EC-WT, whereas they must go to infinity faster than polynomially to get EC-UWT. It seems interesting that WT, UWT, EC-WT and EC-UWT do not depend on $\boldsymbol{b}$, $\boldsymbol{m}$ (with $m_0=1$) and ω.

- **EC-Quasi-Polynomial Tractability**

  APP is EC-QPT if there are positive C and t such that

  $$n(\varepsilon,\mathrm{APP}_s)\le C\,\exp\left(t\,(1+\log\,s)(1+\log\,(1+\log\,\varepsilon^{-1}))\right)\quad\text{for all}\;\;s\in\mathbb{N},\;\varepsilon\in(0,1).$$

  The infimum of t satisfying the bound above is denoted by t^*, and is called the exponent of EC-QPT. Obviously, EC-QPT implies EC-WT.

  Observe that

  $$\begin{aligned}\exp\left(t\,(1+\log\,s)(1+\log\,(1+\log\,\varepsilon^{-1}))\right)&=[\mathrm{e}\,s]^{\,t\,(1+\log\,(1+\log\,\varepsilon^{-1}))}\\&=[\mathrm{e}(1+\log\,\varepsilon^{-1})]^{\,t\,(1+\log\,s)}.\end{aligned}$$

  We will sometimes use these equivalent formulations to establish EC-QPT.

Theorem 8.

APP *is* EC-QPT *iff*

$$m_0 = 1, \quad B^* := \sup_{s\in\mathbb{N}} \frac{\sum_{j=1}^s b_j^{-1}}{1+\log\, s} < \infty, \quad \text{and} \;\; \alpha := \liminf_{j\to\infty} \frac{(1+\log\, j)\,\log\, a_j}{j} > 0.$$

If this holds then the exponent of EC-QPT *satisfies*

$$t^* \in \left[\max\left(B^*, \frac{\log\,(1+m_1)}{\alpha}\right), B^* + \frac{\log\,(1+m_{\max})}{\alpha}\right].$$

In particular, if $\alpha=\infty$ then $t^* = B^*$.

Proof. We first prove that EC-QPT implies the conditions on m_0, B^* and α. Since EC-QPT yields EC-WT, we have $m_0 = 1$. To prove that $B^* < \infty$, we relate EC-QPT to EXP. From

$$n = n(\varepsilon, \mathrm{APP}_s) \le C\, \exp\big(t\,(1+\log\, s)(1+\log\,(1+\log\,\varepsilon^{-1}))\big)$$

we conclude that

$$e(n, \mathrm{APP}_s) \le \varepsilon \le \mathrm{e}\cdot \exp\left(-\frac{1}{\mathrm{e}}\left(\frac{n}{C}\right)^{(t(1+\log\, s))^{-1}}\right).$$

Due to Theorem 1, EXP holds with the exponent $1/B_s = 1/\sum_{j=1}^s b_j^{-1}$. Therefore $1/(t(1+\log\, s)) \le 1/B_s$ and

$$\frac{B_s}{1+\log\, s} \le t \quad \text{for all} \;\; s\in\mathbb{N}.$$

Hence, $B^* < \infty$ and $t \ge B^*$, as claimed.

To prove that $\alpha > 0$, we proceed similarly as for EC-WT and EC-UWT. That is, for a positive δ, we take

$$x(\varepsilon) = (1+\delta)(a_1 + \cdots + a_s) \le (1+\delta)\, s\, a_s.$$

Now (ii) of Lemma 1 yields

$$s\,\log\,(1+m_1) \le \log\, n(\varepsilon, \mathrm{APP}_s) \le \log\, C \; + \; t(1+\log\, s)\,(1+\log\,(1+\log\,\varepsilon^{-1})),$$

where

$$1+\log\,(1+\log\,\varepsilon^{-1})) \le 1+\log\left(1+\frac{(1+\delta)\log\,\omega^{-1}}{2}\, s\, a_s\right).$$

For large s, this proves that

$$\frac{t\,(1+\log\, s)\,\log\, a_s}{s} \ge \log\,(1+m_1) \; + \; o(s).$$

Hence, $\alpha > 0$ and $t \ge \log(1+m_1)/\alpha$, as claimed.

We now prove that $m_0 = 1$, $B^* < \infty$ and $\alpha > 0$ imply EC-QPT.

From $m_0 = 1$ and Lemma 1 we have $n(\varepsilon, \mathrm{APP}_s) = 1$ for $x(\varepsilon) \le a_1$, whereas for $x(\varepsilon) > a_1$, we have

$$\begin{aligned} n(\varepsilon, \mathrm{APP}_s) &\le \prod_{j=1}^{\min(s,j(\varepsilon))} \left(1 + m_{\max} \left(\frac{x(\varepsilon)}{a_j}\right)^{1/b_j}\right) \\ &\le (1+m_{\max})^{\min(s,j(\varepsilon))} \prod_{j=1}^{\min(s,j(\varepsilon))} \left(\frac{x(\varepsilon)}{a_j}\right)^{1/b_j}. \end{aligned}$$

From the definition (12) of $x(\varepsilon)$, we get

$$n(\varepsilon, \mathrm{APP}_s) \le (1+m_{\max})^{\min(s,j(\varepsilon))}\, C_{s,\varepsilon} \left[\mathrm{e}\, \log\, \varepsilon^{-1}\right]^{\sum_{j=1}^{\min(s,j(\varepsilon))} b_j^{-1}}, \tag{24}$$

where

$$C_{s,\varepsilon} = \prod_{j=1}^{\min(s,j(\varepsilon))} \left(\frac{2}{a_j\, \mathrm{e}\, \log\, \omega^{-1}}\right)^{1/b_j}.$$

Note that (24) holds for all $s \in \mathbb{N}$ and all $\varepsilon \in (0,1)$ if we take $j(\varepsilon) = 0$ for $x(\varepsilon) \le a_1$.

We now use the assumption that $\alpha > 0$. This means that for any $\delta \in (0, \alpha)$ there is an integer j_δ such that

$$a_j \ge \exp\left(\frac{\delta\, j}{1 + \log\, j}\right) \qquad \text{for all} \quad j \ge j_\delta.$$

This means that $\lim_j a_j = \infty$, and this convergence is almost exponential in j.

We turn to $j(\varepsilon)$ defined by (16). Now $j(\varepsilon)$ goes to infinity as ε approaches zero. For $\log\, x(\varepsilon) \ge \delta$, i.e., for $\varepsilon \le \omega^{\mathrm{e}^\delta/2}$, we have

$$j(\varepsilon) \le \max(j_\delta, J(\varepsilon)),$$

where $J(\varepsilon)$ is a solution of the nonlinear equation

$$\frac{\log\, x(\varepsilon)}{\delta} = \frac{J(\varepsilon)}{1 + \log\, J(\varepsilon)}. \tag{25}$$

The solution is unique since the function $y/(1 + \log\, y)$ is increasing for $y \ge 1$.

Let $a(\varepsilon) = (\log x(\varepsilon))/\delta$. Then we have from (25) that $J(\varepsilon) = a(\varepsilon)(1 + \log J(\varepsilon))$. Now we write $J(\varepsilon)$ in the form

$$J(\varepsilon) = (1 + f(\varepsilon))a(\varepsilon) \log a(\varepsilon),$$

where $f(\varepsilon)$ is given by

$$f(\varepsilon) = \frac{1 + \log(1 + \log J(\varepsilon))}{\log a(\varepsilon)} = \frac{1 + \frac{1}{\log(1+\log J(\varepsilon))}}{\frac{\log J(\varepsilon)}{\log(1+\log J(\varepsilon))} - 1} = o(1) \quad \text{for} \quad \varepsilon \to 0.$$

Hence we have

$$J(\varepsilon) \;=\; (1 + o(1))a(\varepsilon) \log a(\varepsilon)$$

$$
\begin{aligned}
&= \frac{1+o(1)}{\delta}\,[\log\, x(\varepsilon)]\ \log\,\frac{\log\, x(\varepsilon)}{\delta} \\
&= \frac{1+o(1)}{\delta}\,[\log\,\log\,\varepsilon^{-1}]\ \log\,\log\,\log\,\varepsilon^{-1}. \qquad (26)
\end{aligned}
$$

We turn to (24). Note that $\lim_j a_j = \infty$ implies that only a finite number of factors in $C_{s,\varepsilon}$ is larger than one. Therefore

$$
C_{s,\varepsilon} \le C_1 := \sup_{s\in\mathbb{N},\ \varepsilon\in(0,1)} \prod_{j=1}^{\min(s,j(\varepsilon))} \left(\frac{2}{a_j\,\mathrm{e}\,\log\,\omega^{-1}}\right)^{1/b_j} < \infty.
$$

Furthermore, from the assumption $B^* < \infty$ we have

$$
\sum_{j=1}^{\min(s,j(\varepsilon))} b_j^{-1} = \frac{\sum_{j=1}^{\min(s,j(\varepsilon))} b_j^{-1}}{1+\log\, s}\,(1+\log\, s) \le B^*\,(1+\log\, s).
$$

Therefore we can rewrite (24) as

$$
n(\varepsilon, \mathrm{APP}_s) \le (1+m_{\max})^{\min(s,j(\varepsilon))}\, C_1\,[\mathrm{e}\,(1+\log\,\varepsilon^{-1})]^{B^*\,(1+\log\, s)}. \qquad (27)
$$

We now analyze the first factor $\beta := (1+m_{\max})^{\min(s,j(\varepsilon))}$ in (27). Let $s^* \in \mathbb{N}$ and $\varepsilon^* \in (0,1)$ be arbitrary. Note that for $s \le s^*$ or for $\varepsilon \in [\varepsilon^*, 1]$ we have

$$
\beta \le (1+m_{\max})^{\max(s^*,\, j(\varepsilon^*))} =: C_2 < \infty.
$$

Hence, without loss of generality we can consider

$$
s > s^* \quad \text{and} \quad \varepsilon \in (0, \varepsilon^*).
$$

We now choose s^* such that $s^* \ge j_\delta$. For any positive $\eta \in (0,1)$ we choose a positive ε^* such that for all $\varepsilon \in (0,\varepsilon^*)$ we have

$$
\begin{aligned}
\log\,\log\,\log\,\varepsilon^{-1} &\ge \frac{\delta}{1-\eta}, &(28)\\
\frac{\delta\, J(\varepsilon)}{\log\,\log\,\varepsilon^{-1}\ \log\,\log\,\log\,\varepsilon^{-1}} &\in [1-\eta, 1+\eta], &(29)\\
\frac{(1+\eta)}{\delta\left(1+\frac{\log(1-\eta)/\delta + \log\,\log\,\log\,\log\,\varepsilon^{-1}}{\log\,\log\,\log\,\varepsilon^{-1}}\right)} &\le \frac{1+2\eta}{\delta}. &(30)
\end{aligned}
$$

Observe that such a positive ε^* exists since (28) clearly holds for small ε, whereas (29) holds due to (26), and (30) holds since the limit of the left hand side, as $\varepsilon \to 0$, is $(1+\eta)/\delta$ which is smaller than the right hand side.

We are ready to estimate

$$
\beta = [\mathrm{e}\, s]^{y_{s,\varepsilon}} = [\mathrm{e}\,(1+\log\,\varepsilon^{-1})]^{z_{s,\,\varepsilon}},
$$

where

$$
y_{s,\varepsilon} = \frac{\min(s, j(\varepsilon))\,\log\,(1+m_{\max})}{\log\,(\mathrm{e}s)},
$$

$$z_{s,\varepsilon} \;=\; \frac{\min(s, j(\varepsilon))\, \log(1+m_{\max})}{\log(\mathrm{e}\,(1+\log\,\varepsilon^{-1}))}.$$

We consider two cases depending on whether s or $J(\varepsilon)$ is larger.

Case 1. Assume that $s \le J(\varepsilon)$.

Note that the function $y/(1+\log\, y)$ is an increasing function of $y \in [1,\infty)$. Therefore

$$\frac{s}{1+\log\, s} \le \frac{J(\varepsilon)}{1+\log\, J(\varepsilon)}.$$

Due to (29) and (30),

$$\begin{aligned}\frac{J(\varepsilon)}{1+\log\, J(\varepsilon)} &\le \frac{(1+\eta)\,\log\,\log\,\varepsilon^{-1}\ \log\,\log\,\log\,\varepsilon^{-1}}{\delta(1+\log\frac{1-\eta}{\delta}+\log\,\log\,\log\,\varepsilon^{-1}+\log\,\log\,\log\,\log\,\varepsilon^{-1})}\\ &\le \frac{1+2\eta}{\delta}\,\log\,\log\,\varepsilon^{-1} \le \frac{1+2\eta}{\delta}\,(1+\log(1+\log\,\varepsilon^{-1})).\end{aligned}$$

Hence,

$$\begin{aligned}y_{s,\varepsilon} &\le \frac{s}{1+\log\, s}\,\log(1+m_{\max}) \le \frac{J(\varepsilon)}{1+\log\, J(\varepsilon)}\,\log(1+m_{\max})\\ &\le \frac{(1+2\eta)\,\log(1+m_{\max})}{\delta}\,\big(1+\log(1+\log\,\varepsilon^{-1})\big)\,.\end{aligned}$$

This yields

$$\beta \le [\mathrm{e}s]^{\delta^{-1}(1+2\eta)\,\log(1+m_{\max})\,(1+\log(1+\log\,\varepsilon^{-1}))}$$

which can be equivalently written as

$$\beta \le \exp\left(\left[\frac{1+2\eta}{\delta}\,\log(1+m_{\max})\right]\,(1+\log\, s)(1+\log(1+\log\,\varepsilon^{-1}))\right).$$

This and (27) yield EC-QPT with $t \le B^* + \delta^{-1}(1+2\eta)\,\log(1+m_{\max})$. Since δ can be arbitrarily close to α and η can be arbitrarily small, we conclude that the exponent of EC-QPT in this case satisfies

$$t \le B^* + \frac{\log(1+m_{\max})}{\alpha}.$$

Case 2. Assume that $s > J(\varepsilon)$.

Then $s > \delta^{-1}(1-\eta)\,[\log\,\log\,\varepsilon^{-1}]\ \log\,\log\,\log\,\varepsilon^{-1} \ge \log\,\log\,\varepsilon^{-1}$ due to (28) and (29). Hence, $\log\, s \ge \log\,\log\,\log\,\varepsilon^{-1}$. We now estimate $z_{s,\varepsilon}$. Assume that $\varepsilon > 0$ is small enough such that $j(\varepsilon) \le J(\varepsilon)$. Then we have

$$\begin{aligned}z_{s,\varepsilon} &\le \frac{j(\varepsilon)\,\log(1+m_{\max})}{1+\log(1+\log\,\varepsilon^{-1})} \le \frac{J(\varepsilon)\,\log(1+m_{\max})}{1+\log(1+\log\,\varepsilon^{-1})}\\ &\le \frac{(1+\eta)\,\log(1+m_{\max})}{\delta}\,\frac{\log\,\log\,\varepsilon^{-1}\ \log\,\log\,\log\,\varepsilon^{-1}}{1+\log(1+\log\,\varepsilon^{-1})}\\ &\le \frac{(1+\eta)\,\log(1+m_{\max})}{\delta}\,\log\,\log\,\log\,\varepsilon^{-1}\end{aligned}$$

$$\leq \quad \frac{(1+\eta)\,\log\,(1+m_{\max})}{\delta}\,(1+\log\,s).$$

Hence we have

$$\beta \leq \exp\left(\left[\frac{1+\eta}{\delta}\,\log\,(1+m_{\max})\right]\,(1+\log\,s)(1+\log\,(1+\log\,\varepsilon^{-1}))\right),$$

and the rest of the proof goes like in Case 1. This completes the proof. □

We compare Theorems 4 and 8. The assumption $m_0 = 1$ is needed for both QPT and EC-QPT. However, QPT holds for all $\boldsymbol{a}$ and $\boldsymbol{b}$, whereas for EC-QPT we need to assume that $B^* < \infty$ and $\alpha > 0$. This means that b_j must go to infinity roughly at least like j, and a_j must go to infinity almost exponentially fast.

- **EC-Polynomial and EC-Strong Polynomial Tractability**

APP is EC-PT iff there are positive C, p and $q \geq 0$ such that

$$n(\varepsilon, \mathrm{APP}_s) \leq C\,s^{\,q}\,(1+\log\,\varepsilon^{-1})^p \quad \text{ for all } \; s \in \mathbb{N},\; \varepsilon \in (0,1).$$

APP is EC-SPT if the last bound holds with $q = 0$, and then the infimum of p is denoted by p^*, and is called the exponent of EC-SPT.

Theorem 9.

APP *is* EC-PT *iff* APP *is* EC-SPT *iff*

$$m_0 = 1, \quad B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \quad \textit{and} \quad \alpha^* = \liminf_{j\to\infty} \frac{\log\,a_j}{j} > 0.$$

If these conditions hold then the exponent of EC-SPT *satisfies*

$$p^* \in \left[\max\left(B, \frac{\log(1+m_1)}{\alpha^*}\right), B + \frac{\log(1+m_{\max})}{\alpha^*}\right].$$

In particular, if $\alpha^* = \infty$ then $p^* = B$.

Proof. We prove that EC-PT implies $m_0 = 1$, $B < \infty$ and $\alpha^* > 0$, and then that $m_0 = 1$, $B < \infty$ and $\alpha^* > 0$ imply EC-SPT and find bounds on the exponent of EC-SPT.

EC-PT implies EC-WT and therefore $m_0 = 1$. It is easy to show that EC-PT implies UEXP. Indeed, the bound on EC-PT yields that

$$e(n, \mathrm{APP}_s) \leq \mathrm{e} \cdot \mathrm{e}^{-((n-1)/(C\,s^{\,q}))^{1/p}} \quad \text{ for all } \; n \in \mathbb{N}.$$

Hence, UEXP holds and the exponent of UEXP is at least $1/p$. Then Theorem 1 implies that $B < \infty$, and $p \geq B$.

To prove that $\alpha^* > 0$, we proceed similarly as for EC-WT. That is, for $\delta > 0$ we take $x(\varepsilon) = (1+\delta)(a_1 + \cdots + a_s)$ and then (ii) of Lemma 1 and the bound on EC-PT yield

$$(1+m_1)^s \leq n(\varepsilon, \mathrm{APP}_s) \leq C\,s^{\,q}\left(1 + \frac{(1+\delta)\,\log\,\omega^{-1}}{2}(a_1 + \cdots + a_s)\right)^p.$$

Since $a_1 \le a_2 \le \cdots$, this implies that

$$s a_s \ge a_1 + \cdots + a_s \ge \frac{2}{(1+\delta)\,\log\,\omega^{-1}}\left[\left(\frac{(1+m_1)^s}{C\,s^{\,q}}\right)^{1/p} - 1\right].$$

Hence,

$$\alpha^* = \liminf_{s\to\infty} \frac{\log\,a_s}{s} \ \ge\ \frac{\log(1+m_1)}{p} > 0,$$

as claimed. This also shows that $p \ge \log(1+m_1)/\alpha^*$.

This reasoning also holds for all p for which EC-SPT holds. Therefore the exponent p^* of EC-SPT is at least $p^* \ge \log(1+m_1)/\alpha^*$. Furthermore, p^* cannot be smaller then the reciprocal of the exponent of UEXP, so that $p^* \ge B$. This proves the lower bound on p^*.

We now assume that $m_0 = 1$, $B < \infty$ and $\alpha^* > 0$. Note that $\alpha^* > 0$ means that a_j are exponentially large in j for large j. Indeed, for $\delta \in (0, \alpha^*)$ there is an integer j_δ^* such that

$$a_j \ge \exp(\delta\, j) \quad \text{ for all } \ j \ge j_\delta^*.$$

This yields that for $j(\varepsilon)$ defined by (16) we have

$$j(\varepsilon) \le \max\left(j_\delta^*, \frac{\log\,x(\varepsilon)}{\delta}\right).$$

For $x(\varepsilon) \le a_1$, (i) of Lemma 1 states that $n(\varepsilon, \mathrm{APP}_s) = 1$, whereas for $x(\varepsilon) > a_1$, (iii) of Lemma 1 yields

$$\begin{aligned}
n(\varepsilon, \mathrm{APP}_s) \ &\le \prod_{j=1}^{\min(s,j(\varepsilon))} \left(1 + m_{\max}\left(\frac{x(\varepsilon)}{a_j}\right)^{1/b_j}\right) \\
&\le \left[\prod_{j=1}^{\min(s,j(\varepsilon))} \left(\frac{x(\varepsilon)}{a_j}\right)^{1/b_j}\right] (1+m_{\max})^{\min(s,j(\varepsilon))} \\
&\le \left(\frac{x(\varepsilon)}{a_1}\right)^{B} \max\left((1+m_{\max})^{j_\delta^*}, [x(\varepsilon)]^{(\log(1+m_{\max}))/\delta}\right).
\end{aligned}$$

Since $x(\varepsilon) = \Theta(\log\,\varepsilon^{-1})$ we obtain EC-SPT with $p \le B + (\log(1+m_{\max}))/\delta$. Taking δ arbitrarily close to α^*, we obtain that the exponent of EC-SPT is at most

$$p^* \le B + \frac{\log(1+m_{\max})}{\alpha^*}.$$

This completes the proof. □

We now compare Theorems 5 and 10 for SPT and EC-SPT. In both cases, we have $m_0 = 1$. However, the conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$ are quite different. SPT holds for all $\boldsymbol{b}$, whereas for EC-SPT we must assume that $B < \infty$, i.e., b_j must go to infinity at least like j. The conditions on $\boldsymbol{a}$ are even more striking. SPT holds for a_j going to infinity quite slowly like $\log\,j$, whereas EC-SPT requires that a_j goes exponentially fast to infinity with j.

6 Summary

In the following table we summarize the tractability results. We tabulate the various notions of tractability with their corresponding "if and only if" conditions:

Tractability notion	iff-conditions
(t_1, t_2)-WT	$t_1 > 1$ or $m_0 = 1$
WT, UWT, QPT	$m_0 = 1$
PT, SPT	$m_0 = 1$, and $\lim_j \frac{a_j}{\log j} > 0$
EC-(t_1, t_2)-WT	$t_1 > 1$, or $t_2 > 1$ and $m_0 = 1$
EC-WT	$m_0 = 1$, and $\lim_j a_j = \infty$
EC-UWT	$m_0 = 1$, and $\lim_j \frac{\log a_j}{\log j} = \infty$
EC-QPT	$m_0 = 1$, $\sup_s \frac{\sum_{j=1}^s b_j^{-1}}{1+\log s} < \infty$, and $\liminf_j \frac{(1+\log j)\log a_j}{j} > 0$
EC-PT, EC-SPT	$m_0 = 1$, $\sum_{j=1}^{\infty} b_j^{-1} < \infty$, and $\liminf_j \frac{\log a_j}{j} > 0$

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